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A topologist's view of perfect and acyclic groups

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Introduction

Perfect groups, and in particular non-abelian simple groups, are important objects of study in finite group theory in their own right. In contrast, for a topologist or geometer, perfect groups tend to be those that arise in certain interesting situations, for example via the study of fundamental groups. In the first section below, we try to explain why this might be so. The second section outlines what general results about perfect groups are known, or conjectured. While the author's biases are naturally reflected in the choice of material in these parts, it is in the third that they become rampant. The reader will find there an exhibition of specimens of a key class of perfect groups, namely the acyclic groups.

1 Motivation

1.1 Algebraic motivation

This is very intuitive. The basic point is that abelian groups are much simpler objects to consider than arbitrary groups. For example, finitely generated abelian groups are easily described, since each is the direct product of a finite number of cyclic groups, whose orders therefore determine the finitely generated abelian group (up to isomorphism). On the other hand, as soon as one allows more than one element in a generating set, finitely generated non-abelian groups get wretchedly difficult to describe. For instance, it is known that every countable group embeds in a two-generator simple group, and there are at least $2^{\aleph_0}$ such groups [68, IV.3].

It follows that a good invariant of any group G is its abelianization G_{ab}. This is also known as its first homology group $H_1(G)$ (for homology we use trivial integer coefficients unless otherwise stated), or commutator quotient group G/G'. Here $G' = [G, G]$ is the first derived or commutator subgroup, the subgroup of G generated by all commutators $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. The invariant G_{ab} is characterized by the universal property that every group homomorphism from G to an abelian group factors uniquely through the epimorphism $G \to G_{\mathrm{ab}}$.

The beauty of this invariant is that it enables one to obtain an abelian group from a group that may be truly horrible in its complexity. Now although there is much that can be said (e.g. [40]) about abelian groups, they are indeed easier objects to deal with than arbitrary groups. There is however always a price to be paid for simple invariants. That price is the information lost in passing to the invariant. There are two ways of looking at this situation.

First, the building blocks of the theory will be those groups whose invariant is trivial. Here that means that the group G has trivial abelianization and so is equal to its commutator subgroup. In other words, G is generated by its commutators, and every element of G can be written as a product of commutators. This is the definition of a *perfect* group. In fact, any attempt to describe an arbitrary group G by means of abelian groups inevitably leads to a perfect group (possibly trivial), as we shall now see.

In the general case, the information lost in abelianizing is measured by the kernel G' of the epimorphism $G \to G_{\mathrm{ab}}$. This poses the problem of describing the first derived group G'. Since abelianization was considered a good method for attacking the original group, it should also be good enough for the group G'. How much information is lost in passing to its abelianization? This is given by its commutator subgroup, the second derived subgroup

$$G^{(2)} = (G')' = [G', G'].$$

And so on ... In this way one obtains the *derived series*

$$G = G^{(0)} \supseteq G' = G^{(1)} \supseteq G^{(2)} \supseteq \cdots$$

with $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, spinning off an abelian group $G^{(n)}_{\mathrm{ab}}$ at each stage. This process either reaches a perfect group and terminates:

$$G^{(n)} \text{ perfect} \quad \Longleftrightarrow \quad G^{(n+1)} = G^{(n)};$$

or it doesn't. In the latter case there is a new subgroup $G^{(\omega)} = \bigcap_{n \geq 0} G^{(n)}$, and one may apply the whole procedure to it. This leads to the *transfinite derived series* for G, with $G^{(\beta)}$ defined for each ordinal β as follows. If β is the successor of an ordinal α, then define $G^{(\beta)} = [G^{(\alpha)}, G^{(\alpha)}]$. If, on the other hand, β is not a successor ordinal, then put $G^{(\beta)} = \bigcap_{\gamma < \beta} G^{(\gamma)}$. Now, since G is after all a set, its cardinality gives an upper bound for how far this series can descend. Eventually it must reach an ordinal ν for which $G^{(\nu+1)} = G^{(\nu)}$. In other words, it terminates at the (possibly trivial) perfect group $G^{(\nu)}$, the intersection of the transfinite derived series of G.

Hence we have obtained a whole chain of invariants of an arbitrary group G. All except the last are abelian subquotient groups of G, while the last is a perfect subgroup of G. Now the knowledge of all these groups need not allow total reconstruction of the original group, since the groups G' and G_{ab} may fit into a group extension

$$G' \rightarrowtail H \twoheadrightarrow G_{\mathrm{ab}}$$

with G' isomorphic to the kernel of a group epimorphism $H \twoheadrightarrow G_{\text{ab}}$ for some group H that is not isomorphic to G. But that is not the point. The point is that the simple process of extracting all the abelian invariants from a given group G inevitably leads to a perfect subgroup of G that is just as fundamental to a description of G as say the most obvious abelian group, G_{ab}.

1.2 Geometric motivation

The above fact that for any discrete group G, $H_1(G) = G_{\text{ab}}$, has the important topological generalization that for any topological space X

$$H_1(X) = \pi_1(X)_{\text{ab}}.$$

The case where X is the classifying space of G, the Eilenberg–Mac Lane space $K(G,1)$, reduces to

$$H_1(G) = H_1(K(G,1)) = \pi_1(K(G,1))_{\text{ab}} = G_{\text{ab}}.$$

Then the above considerations suggest that for classes of spaces X for which homology groups are easily calculated, perfect subgroups of the fundamental group have a role to play in calculation of $\pi_1(X)$. As an extreme case, if it is known that $H_1(X)$ is the zero group, then $\pi_1(X)$ is perfect. So perfect groups arise as, for example, the fundamental groups of homology spheres (see (2.4.2) below).

For relatively easy visualization, let's do some low-dimensional topology. To see which surfaces have perfect fundamental groups, we start by looking for commutators. For a simple loop on a surface to represent a commutator, it must bound a punctured torus (also known as a bridged annulus [46, p. 20]), as follows.

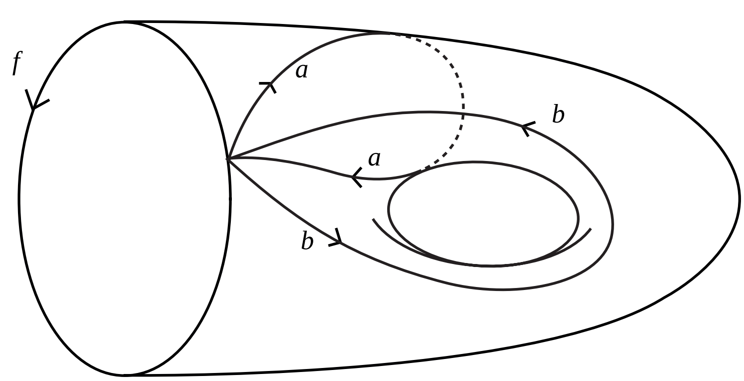

Fig. 1 Punctured torus

The fact that the boundary curve of a punctured torus is indeed a commutator is perhaps most easily checked on the model for the quotient space.

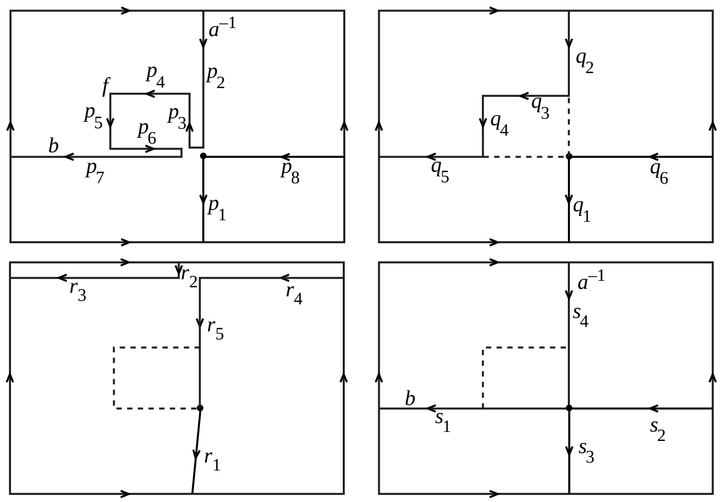

Fig. 2 Homotopy

Here

$$\begin{aligned} a^{-1}fb &= (p_1p_2)(p_3p_4p_5p_6)(p_7p_8) \\ &\simeq q_1q_2q_3q_4q_5q_6 \\ &\simeq r_1r_2r_3r_4r_5 \\ &\simeq (s_1s_2)(s_3s_4) = ba^{-1}. \end{aligned}$$

Hence

$$[f] = [a][b][a]^{-1}[b]^{-1}.$$

This argument is probably more familiar to readers in the form where the puncture that the loop f bounds is filled in. Then f becomes trivial and the proof shows that the classes of a and b commute in the fundamental group of the torus.

More generally, when a simple loop on a surface represents a product of commutators, it bounds a disk-with-handles, as in the figure below (from [26]).

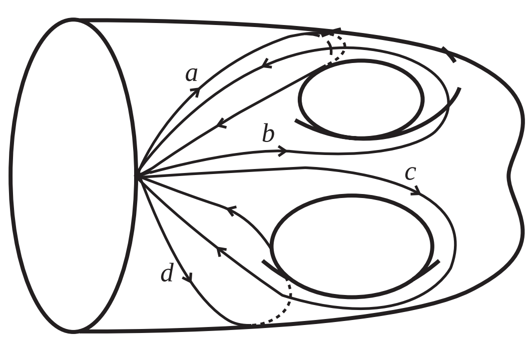

Fig. 3

Now, in the situation of a perfect fundamental group, every element that features in a product of commutators is itself equal to a product of commutators. This can be achieved by making each non-trivial loop such as a, b, c, d in turn bound a disk-with-handles. This procedure has to be iterated infinitely often, resulting in the concept of a *grope* [90], as illustrated below.

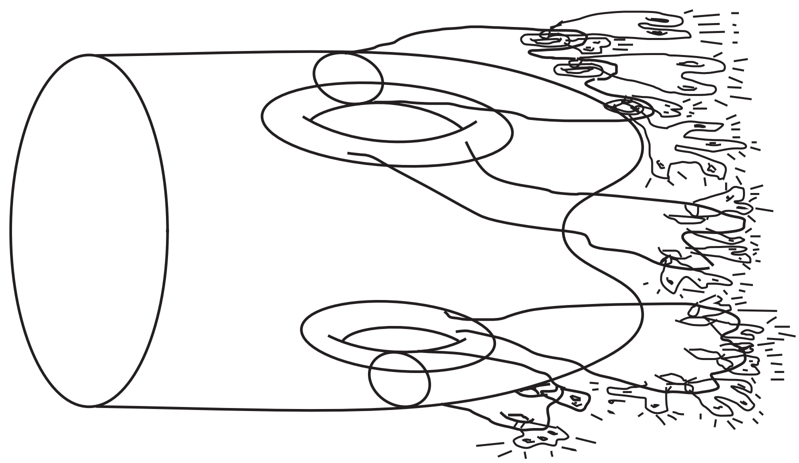

Fig. 4

The formal description of a grope [31] is the direct limit L of a nested sequence of compact 2-dimensional polyhedra

$$L_0 \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \cdots$$

obtained as follows. Let S_g denote an oriented compact surface of positive genus g from which an open disk has been deleted. (So $g = 1$ gives a punctured torus.) Take L_0 as some S_g. To form L_{n+1} from L_n, for each loop a in L_n that generates the group $H_1(L_n)$, attach to L_n some S_{g_a} by identifying the boundary of S_{g_a} with the loop a. Since the fundamental group of S_g punctured is a free group on $2g$ generators, this procedure embeds each $\pi_1(L_n)$, and thus each finitely generated subgroup of $\pi_1(L)$, as a subgroup of a free group, with each generator a of $\pi_1(L_n)$ becoming a product of g_a commutators in $\pi_1(L_{n+1})$. Hence $\pi_1(L)$ is a countable, perfect, locally free group.

The most economical example is the *minimal grope* M^*, for which one takes only one genus one surface at each step. Homotopically, each L_n is in this case just a bouquet of finitely many circles. So M^* is the classifying space of its fundamental group, which is easily seen to have the following description. Let Σ denote the set of all (non-empty) words of finite length on the two symbols $0, 1$. Then $\pi_1(M^*)$ is the group generated by symbols x_w for each $w \in \Sigma$, subject only to all relations $x_w = [x_{w0}, x_{w1}]$. Thus the x_w with w of length n generate the free group $\pi_1(L_n)$, but each is a commutator when embedded in the larger free group $\pi_1(L_{n+1})$.

The 3-dimensional version of this construction occurs in [39, Theorem 2], where a nested sequence of handle-bodies leads to a locally free fundamental group, and conversely.

Aside. Readers of the stimulating material in [26] on perfect groups and geometry should be aware that the wild group $\omega(G)$ of [26, Supplement 12], while there claimed to be the intersection of the transfinite derived series as above (hence perfect), is actually defined in [25] to be the intersection of the series

$$G = G^{[0]} \supseteq G^{[1]} \supseteq G^{[2]} \supseteq \cdots$$

where each $G^{[n+1]}$ is the subgroup of $G^{[n]}$ generated by squares of elements. Since for example

$$aba^{-1}b^{-1} = (aba^{-1})^2 a^2 (a^{-1}b^{-1})^2,$$

certainly $G^{(n)} \subseteq G^{[n]}$. However, there is in general no guarantee that $\omega(G) = \bigcap_{n=0}^{\infty} G^{[n]}$ is perfect. For instance, when G is a group of odd order, $\omega(G) = G$ is soluble.

1.3 Topological motivation

There is in algebraic topology an important theorem of Kan and Thurston that represents homotopy types of spaces as pairs consisting of a discrete group and a distinguished perfect normal subgroup [61]. The key to this representation is Quillen's *plus-construction*, invented to build higher K-theory of rings [79]. We work in **Ho**, the pointed homotopy category of connected CW-complexes. (One can think of this as the topological category of based, connected CW-complexes (called *spaces* below) in which a formal inverse is adjoined to each based homotopy equivalence.) For each space X and perfect normal subgroup N of $\pi_1(X)$, the (relative) plus-construction is a map $q : X \to X_N^+$ that is *acyclic* (meaning that its homotopy fibre F_q is an acyclic space: $\tilde{H}_*(F_q) = 0$) and has the following universal property [9, 5.2].

Lemma 1.3.1 *Let X be any connected space. If N is a perfect normal subgroup of $\pi_1(X)$, then any map $f : X \to Y$ such that $f_*(N)$ is trivial factors through the plus-construction $q : X \to X_N^+$, uniquely up to homotopy under X.*

The Kan–Thurston theorem states that any space Y has the homotopy type of some X_N^+, where X may be chosen to be the classifying space of a discrete group T. Thus the group T and perfect normal subgroup N of T determine Y as BT_N^+. In fact, one can go further and show that the plus-construction induces an equivalence of categories. The codomain category is **Ho**. The domain is obtained from the category of pairs (G, N) of group G and perfect normal subgroup N as a category of fractions, by formally inverting those group homomorphisms that induce isomorphisms of the quotient G/N and also induce isomorphisms of G in homology with coefficients induced from G/N [6]. This equivalence of categories means that many notions and constructions in homotopy theory have parallels in group theory, in a way that crucially involves perfect groups [53].

1.4 Examples

Where are good places to go hunting for perfect groups?

A group theorist would instantly point out that any non-abelian simple group G must be perfect, since the normal subgroup G' cannot be trivial. Thus the great corpus of knowledge of finite simple groups provides information about finite perfect groups [56], although perfectness has not been a key concept in the classification of finite simple groups.

Many finite simple groups occur as groups of matrices over finite fields, for example, matrices of determinant 1. (Actually, this gives a perfect group; one has to divide out by the centre, comprising the scalar matrices, to obtain a simple group.) An $n \times n$ matrix may be regarded as an $(n+1) \times (n+1)$ matrix by adjoining an entry 1 in the $(n+1, n+1)$ position, and zeroes elsewhere in the $(n+1)$-st row and column. In this way we can take the union of all the groups of determinant 1 matrices to obtain the infinite special linear group. Note that this group is centreless, for the only infinite scalar matrix that has some diagonal entries equal to 1 is the identity matrix. For about a hundred years it has been known that this group is simple, whenever the entries lie in a field; moreover the field need not be finite [80, 3.2]. These results are essentially algebraic, although much of their motivation came from projective geometry, in particular groups of collineations of projective spaces (e.g. [67, VII.5], [51, 7.1.1]).

Combinatorial topology provided an essential input that led to the generalization of these examples from fields to arbitrary rings, and thus gave birth to the subject of algebraic K-theory. This developed from J.H.C. Whitehead's attempt [97], [98] to describe a homotopy equivalence of compact spaces as an iteration of elementary collapses and expansions (a map homotopic to such an iteration being called a *simple homotopy equivalence*). Each elementary collapse/expansion is determined by the attaching map of an attaching r-cell. (It was for this purpose that CW-complexes were invented.) The complete endeavour is worthy of a book ([28] is unhesitatingly recommended) and too technical even to outline in the space available here. However, some informal remarks hint at how it inexorably leads to an important class of perfect groups.

First, by means of mapping cylinders one manœuvres into the situation of a finite r-dimensional complex K, with subcomplex L which is a strong deformation retract of K. Moreover, by a geometric normalization argument ('trading cells') one may assume that K is determined from a subcomplex K' by n attaching maps $f_i : S^{r-1} \to K'$, and K' in turn is formed from L by the attachment of n $(r-1)$-cells, via maps $g_i : S^{r-2} \to L$. Now the strong deformation retraction kills the first and last groups in the homotopy exact sequence of the triple (K, K', L):

$$\pi_r(K, L) \to \pi_r(K, K') \xrightarrow{\partial} \pi_{r-1}(K', L) \to \pi_{r-1}(K, L).$$

Thus the connecting homomorphism ∂ is an isomorphism. This is an isomorphism of $\mathbb{Z}[\pi_1(L)]$-modules because the sequence respects the natural action of $\pi = \pi_1(L)$ on the relative homotopy groups [88, 7.3.10]. (Recall that when a group G acts on an abelian group A compatibly with addition in A, then A becomes a module over the *group ring* $\mathbb{Z}[G]$. $\mathbb{Z}[G]$ consists of finite formal sums $\sum m_i x_i$ $(m_i \in \mathbb{Z},\ x_i \in G)$ with

$$\left(\sum m_i x_i\right) \cdot \left(\sum n_j y_j\right) = \sum_{i,j} m_i n_j x_i y_j,$$

as well as the obvious formal addition.) In the present setup, both $\pi_r(K, K')$ and $\pi_{r-1}(K', L)$ are free $\mathbb{Z}[\pi]$-modules (isomorphic to the direct sum of n copies of $\mathbb{Z}[\pi]$), with generating classes represented by the f_i and g_i respectively. Hence ∂ defines an $n \times n$ change-of-basis matrix, an element of the group $\mathrm{GL}_n(\mathbb{Z}[\pi])$ of invertible $n \times n$ matrices over the group ring.

What drives the whole theory is the effect of elementary collapses and expansions on this matrix. They correspond variously to products of the following operations:

(i) embedding $\mathrm{GL}_n(\mathbb{Z}[\pi])$ in $\mathrm{GL}_{n+1}(\mathbb{Z}[\pi])$, by adjunction of 1 in the $(n+1, n+1)$-slot, as previously described;

(ii) multiplication by an elementary matrix $e_{ij}(r)$ (an identity matrix, save for a unique off-diagonal entry r in the (i, j)-slot), corresponding to the operation of adding a multiple of one row or column to another;

(iii) multiplication by an invertible diagonal matrix (diagonal entries comprising elements of π and their negatives, these necessarily being units of $\mathbb{Z}[\pi]$).

So, to decide whether a given homotopy equivalence is a simple homotopy equivalence one has to determine whether a matrix in $\mathrm{GL}_n(\mathbb{Z}[\pi])$ is reducible to the identity matrix by a sequence of the above operations. Now define, for any ring R (associative, with identity 1), the subgroup $E_n R$ of $\mathrm{GL}_n R$ to be that generated by all $n \times n$ elementary matrices. Combining (i) and (ii) above gives the subgroup ER of $\mathrm{GL}R$.

An important lemma (the Whitehead lemma) asserts that ER is the commutator subgroup of $\mathrm{GL}R$. Therefore the group $K_1 R := \mathrm{GL}R/ER$ is abelian. Hence the obstruction to a homotopy equivalence being a simple homotopy equivalence (the Whitehead torsion) lies in the abelian group (the *Whitehead group*) $\mathrm{Wh}(\pi)$ obtained from $K_1(\mathbb{Z}[\pi])$ by factoring out the subgroup of diagonal matrices specified in (iii) above. Conversely, every element of $\mathrm{Wh}(\pi)$ is geometrically realizable as such an obstruction. The calculation of $\mathrm{Wh}(\pi)$, even for finite π, is an active area of research [78].

In general, K_1R is but one of a whole sequence of abelian group invariants of a ring R, whose study comprises the topic of algebraic K-theory [5], [9], [81]. The group ER plays a crucial role in this subject. Since for distinct i, j, k

$$e_{ij}(r) = [e_{ik}(r), e_{kj}(1)] \tag{1.4.1}$$

each E_nR, $n \geq 3$, is generated by commutators and so perfect. Hence ER too is perfect, and indeed is the intersection of the transfinite derived series of $\mathrm{GL}R$. We referred above to Quillen's use of the plus-construction to define higher K-groups of a ring R. Specifically, for $i \geq 1$, he defined K_iR to be the ith homotopy group of the space $B\mathrm{GL}R^+_{ER}$. It follows from Lemma 1.3.1 that for $i = 1$ this agrees with the previous definition of K_1R.

The observation that ER is perfect can be pushed further. For any ideal I of R, EI is defined to be the subgroup of ER generated by all $e_{ij}(a)$ with $a \in I$, and $E(R, I)$ has generators of the form $\alpha e_{ij}(a)\alpha^{-1}$ where $\alpha \in ER$ and $e_{ij}(a) \in EI$. It is readily shown to be normal in $\mathrm{GL}R$. Now suppose that I is an idempotent ideal. This means that we can write

$$a = \sum_p a'_p a''_p,$$

a finite sum with each $a'_p, a''_p \in I$. So for any k distinct from both i and j, as in Equation 1.4.1

$$\begin{aligned} \alpha e_{ij}(a)\alpha^{-1} &= \textstyle\prod_p [\alpha e_{ik}(a'_p)\alpha^{-1}, \alpha e_{kj}(a''_p)\alpha^{-1}] \\ &\in [E(R, I), E(R, I)]. \end{aligned}$$

This makes $E(R, I)$ a perfect normal subgroup of ER. With more work one can show the converse, and thereby characterize the perfect normal subgroups of $\mathrm{GL}R$ as precisely those $E(R, I)$ with I an idempotent ideal of R [15]. In favourable cases, such as R being a commutative domain, it is known that there are no proper non-zero idempotent ideals, making ER the sole non-trivial perfect normal subgroup of $\mathrm{GL}R$.

2 Basic facts

2.1 Preliminaries

If all the general theorems on perfect groups were laid end-to-end, they would scarcely reach a conclusion. What we compile here amounts to little more than a collection of observations, which at least makes for easy reading.

Since perfect groups are those generated by their commutators, and the homomorphic image of a commutator is again a commutator, any homomorphic image of a perfect group must also be perfect. In the other direction, let

$$N \hookrightarrow G \twoheadrightarrow Q$$

be a group extension with both N and Q perfect. Then the associated homology exact sequence

$$N/[N, G] \to H_1(G) \to H_1(Q)$$

has first and last terms zero, leaving G perfect. (The alternative, barehanded calculation is a simple exercise.) So the class of perfect groups is extension-closed. Evidently it is also closed under the formation of finite direct and free products. Further, the direct limit of a direct system of perfect groups is perfect too, since every element in the limit can be traced back to a product of commutators somewhere.

If one restricts to subgroups of a given group G, then again all conjugates and products of perfect subgroups are perfect. So the product of all perfect subgroups is both normal and perfect. It must contain every perfect subgroup of G, and so is the *maximum perfect subgroup*, or *perfect radical*, $\mathcal{P}G$ of G. Because whenever $\mathcal{P}G \leq H$, a subgroup of G,

$$\mathcal{P}G = [\mathcal{P}G, \mathcal{P}G] \leq [H, H],$$

$\mathcal{P}G$ must lie inside each term of the derived series, and then within each term of the transfinite derived series of G. Thus $\mathcal{P}G$ is contained in the intersection of the transfinite derived series. However, we have already noted that this intersection is itself perfect, hence necessarily contained in $\mathcal{P}G$. So we have an alternative description of $\mathcal{P}G$ as the intersection of the transfinite derived series of G. Its elements are thereby those elements of G that can be expressed as products of commutators of elements that are themselves products of commutators of elements that are in turn ..., and so (transfinitely) on. Since the image of a perfect group is perfect, any homomorphism $G \to H$ maps $\mathcal{P}G$ inside $\mathcal{P}H$, so the construction $G \mapsto \mathcal{P}G$ is functorial.

2.2 Actions of perfect groups

Just as groups with trivial first derived group are called abelian, those with trivially intersecting transfinite derived series are sometimes called *hypoabelian.* So certainly soluble groups are hypoabelian. Abelianization is a universal construction for obtaining an abelian quotient of an arbitrary group. Likewise there is a process that might be termed *hypoabelianization* (by those who can tolerate a mixture of Greek, Norwegian and Latin in a single word). For, let P be a subgroup of G that has perfect image $\overline{P}$ in the quotient $G/\mathcal{P}G$. After multiplying by $\mathcal{P}G$, we may assume that P contains $\mathcal{P}G$. Then the group extension

$$\mathcal{P}G \hookrightarrow P \twoheadrightarrow \overline{P}$$

forces P to be a perfect subgroup of G. Maximality of $\mathcal{P}G$ makes P equal to $\mathcal{P}G$, and so $\overline{P}$ is trivial. On the other hand, any map from G to a group H with $\mathcal{P}H = 1$ must kill $\mathcal{P}G$ and so factor through $G/\mathcal{P}G$. Therefore the epimorphism $G \twoheadrightarrow G/\mathcal{P}G$ has the universal property of being initial in the category of all maps from G to hypoabelian groups. An example is given by the plus-construction $X \to X^+ = X^+_{\mathcal{P}\pi_1(X)}$ with respect to the maximum perfect subgroup of $\pi_1(X)$. It follows from (1.3.1) that on fundamental groups it induces the hypoabelianization $\pi_1(X) \twoheadrightarrow \pi_1(X)/\mathcal{P}\pi_1(X)$.

A much-used corollary of the above is that any homomorphism from a perfect group to a hypoabelian (for example, soluble) group is trivial. Another 'industrial lemma' is as follows. (Recall the notations $[H, K]$ for the subgroup generated by all commutators $[h, k]$ with $h \in H,\ k \in K,$ and ${}^h k = hkh^{-1}$.)

Lemma 2.2.1 *Let N, P be subgroups of a group G, and suppose that P is perfect. If $[[N, P], P] = 1,$ then $[N, P] = 1$.*

Proof From the Hall–Witt identity

$$ {}^a[[a^{-1}, b], c] \cdot {}^c[[c^{-1}, a], b] \cdot {}^b[[b^{-1}, c], a] = 1 $$

(a brute-force calculation), one has the Three Subgroup lemma: For any three subgroups N, P, Q of a group $G,\ [[P, Q], N]$ lies in every normal subgroup of G that contains both $[[N, P], Q]$ and $[[N, Q], P]$. Here we put $Q = P = [P, P]$.

This forms the basis for an induction argument. When P is a group that acts on a group N, then N and P may both be taken to be subgroups of their semidirect product $G = N \rtimes P$. One then calls the original action of P on N *trivial* if $[N, P] = 1$ in G and *nilpotent* if some

$$ [[\cdots[[N, P], P]\cdots], P] = 1. $$

The extended argument thus shows that if a perfect group acts nilpotently on another group, then it acts trivially.

This result has implications for the study of fibrations in algebraic topology. A fibration $F \to E \to B$ of path-connected spaces is called *orientable* if $\pi_1 B$ acts trivially on the homology groups of F, and *quasi-nilpotent* if the induced action is nilpotent. Normally one spends some effort in showing that theorems about orientable fibrations extend to quasi-nilpotent fibrations too. However, when the fundamental group of B is perfect, we have just seen that the extension is vacuous.

Staying with the group theory, it is interesting to focus on the case where N is a subgroup of P. Our result may be recast as the assertion that if N lies in the hypercentre (the union of the upper central series) of P, then N is contained in the centre $\mathcal{Z}(P)$ of P. Thus the group $P/\mathcal{Z}(P)$ is centreless. In fact, the theory of central subgroups of perfect groups is both simple and elegant.

2.3 Central extensions

Readers may like to treat the following assertions as exercises. Much of the material may be found in [76, ch. 5], proved by elementary means. For an alternative approach by easy obstruction theory, see [9, ch. 8], using the fact that a group extension

$$N \hookrightarrow G \twoheadrightarrow Q$$

is central ($N \subseteq \mathcal{Z}(G)$) if and only if the associated fibration

$$BN \to BG \to BQ$$

is principal. For many related topics, see [22], [47]. We also refer to the epimorphism $G \twoheadrightarrow Q$ (and occasionally G itself) as a central extension (over Q).

Lemma 2.3.1 *A composite of central extensions over perfect groups is a central extension.*

Lemma 2.3.2 *A group G is perfect if and only if for each homomorphism $\phi : G \to R$ and each central extension $\pi : H \twoheadrightarrow R$ there is at most one lift $\mu : G \to H$ with $\phi = \pi\mu$.*

Lemma 2.3.3 *If $G \twoheadrightarrow Q$ is a central extension with Q perfect, then $G' = [G, G]$ is perfect.*

Theorem 2.3.4 *The category of central extensions over a group Q, with morphisms as commuting triangles over Q, has an initial object (the* universal central extension) *if and only if Q is perfect.*

Lemma 2.3.5 *For Q perfect with universal central extension $S \twoheadrightarrow Q$, and any central extension $G \twoheadrightarrow Q$, the group G is perfect if and only if the unique map $S \to G$ over Q is surjective.*

Theorem 2.3.6 *Let $M \hookrightarrow S \twoheadrightarrow Q$ be a central extension. Then the following are equivalent.*

(i) *$S \twoheadrightarrow Q$ is a universal central extension.*
(ii) *S is* superperfect: $H_1(S) = H_2(S) = 0$.
(iii) *S is perfect and every central extension over S splits.*
(iv) *S is perfect and M is isomorphic to $H_2(Q)$, the* Schur multiplicator *of Q.*
(v) *Any group extension $R \hookrightarrow F \twoheadrightarrow Q$ with free group F induces an isomorphism of exact sequences*

$$\begin{array}{ccccc} (R \cap [F,F])/[R,F] & \hookrightarrow & [F,F]/[R,F] & \twoheadrightarrow & Q \\ \downarrow \cong & & \downarrow \cong & & \downarrow \mathrm{id} \\ M & \hookrightarrow & S & \twoheadrightarrow & Q \end{array}$$

For a pleasing analogous theory of extensions of S-modules when S is superperfect, see [23].

It turns out that every abelian group M is the Schur multiplicator of some perfect group Q as in this theorem; moreover, for this purpose S can be chosen (functorially) so that all its homology groups vanish [10] (see (3.1.13) below).

The smallest non-trivial perfect group is the simple group of order 60. Formulating it as the projective special linear group $\mathrm{PSL}_2(\mathbb{F}_5)$, one sees that the universal central extension is its double cover $\mathrm{SL}_2(\mathbb{F}_5)$, the binary icosahedral group. Thus the Schur multiplicator has order 2. Analogous statements hold for other finite simple groups of Lie type. Analysis of this situation led Steinberg [91] to construct, for a ring R, the *Steinberg group* $\mathrm{St}R$ (by mimicking the generators and relations, such as Equation 1.4.1, in ER). Then $\mathrm{St}R \twoheadrightarrow ER$ is the universal central extension of ER [65], whose kernel $H_2(ER)$ became the definition of $K_2(R)$ [76, ch. 5].

A more challenging exercise, based on the 5-term homology exact sequence of a group extension, is to show that if $N \hookrightarrow G \twoheadrightarrow Q$ is a (not necessarily central) extension with G perfect-by-soluble (in other words, $\mathcal{P}G = G^{(n)}$ for some finite n), then there is an exact sequence

$$H_2(\mathcal{P}G) \to H_2(\mathcal{P}Q) \to N/[\mathcal{P}G, N \cap \mathcal{P}G] \to G/\mathcal{P}G \twoheadrightarrow Q/\mathcal{P}Q.$$

2.4 Some open problems

2.4.1 Kervaire conjecture. In characterizing those groups that are the fundamental group of the complement of a smoothly embedded n-sphere in $\mathbb{R}^{n+2}$ (called a *higher knot group*), Kervaire [63] found a necessary condition to be that the group is generated by the conjugates of a single element and its inverse. He was led to conjecture that this condition is always violated by the free product $G * C_\infty$ of a non-trivial group G with the infinite cyclic group C_∞. In this event, let us call the group G *Kervaire.* Evidently, for G to fail to be Kervaire, the abelianization of $G * C_\infty$ must be cyclic, so that G is perfect. (Indeed, it follows from [63] that a superperfect group G is non-Kervaire precisely when $G * C_\infty$ is a higher knot group.)

The few direct assaults on this problem have yielded as Kervaire all locally residually finite [82] and locally indicable groups [58] ('locally indicable' means that every finitely generated subgroup maps onto C_∞), all groups having a faithful finite-dimensional unitary representation [68, p. 50], all torsion-free groups [66], and all groups having a quotient group that is Kervaire [20]. Because confirmation of the conjecture is equivalent to its verification for algebraically closed groups [20], it may be considered solely as a problem in combinatorial group theory, where one asks whether a system of equations in a group admits a solution in some larger group [60]. However, it gains interest from its relation to a number of problems in low-dimensional topology [24], [59], where it is related to other conjectures [41], [89]. One such, discussed in [57], is a problem posed by J.H.C. Whitehead: Is every subcomplex of an aspherical 2-complex also aspherical? The fundamental group of any counterexample must contain both a superperfect and a finitely generated perfect (non-trivial) subgroup.

2.4.2 Homology spheres. We have already noted that the fundamental group π of a homology n-sphere (say, a smooth n-dimensional manifold M with $H_*(M) \cong H_*(S^n)$) must have $H_1(\pi) = 0$. By standard classification of surfaces, we may assume that $n \geq 3$. Since $H_2(\pi)$ is the cokernel of the Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M)$, it too is zero, making π a superperfect, finitely presented group. In [64], Kervaire uses surgery to show that these necessary conditions also suffice whenever $n \geq 5$. This is not the case for $n = 3$, since there the only non-trivial finite group π is the binary icosahedral group $\mathrm{SL}_2(\mathbb{F}_5)$, the fundamental group of the Poincaré 3-sphere. In low dimensions there are the following implications. The first may be deduced from [33], while the others are in [64].

Theorem 2.4.3 *Among the following statements,*

$$\text{(i)} \quad \Longrightarrow \quad \text{(ii)} \quad \Longrightarrow \quad \text{(iii)} \quad \Longrightarrow \quad \text{(iv)}$$

(i) π is the fundamental group of a homology 3-sphere.

(ii) π is a perfect group and has a finite presentation with an equal number of generators and relations.

(iii) π is the fundamental group of a homology 4-sphere.

(iv) For each $n \geqslant 5$, π is the fundamental group of a homology n-sphere.

In [52], it is shown that the implication (iii) $\Rightarrow$ (iv) above is strict.*

2.4.4 μ-problem. For an arbitrary group G, we can define $\mu(G)$ to be the smallest cardinality of a non-empty subset X whose normal closure (the subgroup generated by all conjugates of elements of X and their inverses) is G. This notation is compatible with that of ring theorists. However, there is no convention. For example, in [63], μ is called the weight w. It is shown there that, for $n \geq 1$, when N is a connected n-dimensional submanifold of a simply-connected $(n+2)$-dimensional smooth manifold M, then $\mu(\pi_1(M - N)) = 1$. (Another notation, attributed to P. Hall, is $d_G(G)$, referring to the number of generators of G as a G-group, the G-operation being conjugation.)

Evidently, for any quotient Q of G, $\mu(G) \geq \mu(Q)$. The most interesting comparison is when $Q = G_{\mathrm{ab}}$, since $\mu(G_{\mathrm{ab}})$ is just the rank of the abelian group G_{ab}. The μ-problem asks when equality holds. Let us call

$$\mu_{\mathrm{def}}(G) = \mu(G) - \mu(G_{\mathrm{ab}})$$

the *μ-defect* of G. Since a perfect group P has $\mu(P_{\mathrm{ab}}) = 1$, perfect groups are a promising place to look for positive μ-defects. For example, for any non-trivial perfect Kervaire group P, the free product $C_\infty * P$ has positive μ-defect. In fact, when P has finite composition length, then by Theorem 2.4.5 below, $\mu(P) = 1$, so that $\mu_{\mathrm{def}}(C_\infty * P) = 2 - 1 = 1$. For more extreme examples, the

* *Added in proof:* The strictness of (i) $\Rightarrow$ (ii) follows from C.M. Campbell and E.F. Robertson: A deficiency 0 presentation of $\mathrm{SL}(2, p)$, *Bull. London Math. Soc.* **12** (1980), 17–20. For the irreversibility of (ii) $\Rightarrow$ (iii), see: J.A. Hillman: An homology 4-sphere group with negative deficiency, to appear in *L'Enseignement Math.*

perfect, locally nilpotent groups of (3.1.13) below have infinite μ-defect, as does the countable, locally finite, superperfect group of [19] that has all subnormal subgroups normal, of infinite index.

On the other hand, the same logic suggests that, to find groups with zero μ-defect one should consider groups that are far from perfect on passage to quotients. This leads to the class of *imperfect* groups, namely those groups with no non-trivial perfect quotient. Examples are soluble groups and finite symmetric groups. This class relates to the μ-problem as follows.

Theorem 2.4.5 [19] *If G is an extension of an imperfect group by a group with finite composition length, then $\mu_{\text{def}}(G) = 0$.*

Many of the groups of homeomorphisms referred to in (3.1.6) below also have zero μ-defect. Indeed, much of the literature on these groups is devoted to establishing uniform bounds on the number of conjugates in the expression of an arbitrary element as a product of conjugates of a given element.

2.4.6 Number of commutators. Since in a perfect group G every element is a product of commutators, a very obvious problem, related to the above, is to ask how few commutators are required. The number may be unbounded, as for example, in the case of $E_n(\mathbb{C}[x])$ with $n \geq 3$ [30]. When there is a bound, one writes $c(G)$ for the minimum number of commutators in which any element of G' is expressible.

Linear groups have been the most studied from this point of view. For example, for any field F, $c(E_nF) \leq 2$ and $c(EF) = 1$ [93], $c(E_nR) \leq 5$ for a commutative ring R of stable rank 1 (such as $R = \mathbb{Z}/k$) [30], while for any R $c(ER) \leq 2$ [49].

There is an old conjecture of O. Ore that every finite simple group G has $c(G) = 1$. The binate groups encountered in 3 below all have $c = 1$. Much of the literature on non-binate homeomorphism groups aims to establish bounds on c. For example, the group of piecewise linear homeomorphisms of $\mathbb{R}$ with compact support is perfect with $c \leq 2$ [92].

2.4.7 Finite groups. There is of course a large number of conjectures and open problems concerning finite perfect groups. I mention here two that relate to the material in this chapter. The first asserts that perfect finite groups are rare, and thus stands in contrast to a result cited in (3.1.8) below. Let p_n and g_n denote the number of isomorphism classes of perfect, respectively all, groups of order n.

Conjecture 2.4.8 *For the multiplicative directed set structure on $\mathbb{N}$ where $d \preceq n \Leftrightarrow d \mid n$,*

$$\operatorname{dirlim}(p_n/g_n) = 0.$$

Strong supporting evidence has appeared in [56].

For the second, we call a group G *k-connected* (*k-acyclic* in [23]) if, for $1 \leq i \leq k$, $H_i(G) = 0$ (equivalently the space BG^+ is k-connected). The search for 3-connected finite groups culminated in Milgram's computation [75] that the Mathieu group M_{23} is 4-connected. This still leaves open the following. (Related material is surveyed in [1].)

Problem 2.4.9 Does there exist k such that no non-trivial finite group G is k-connected?

3 Acyclic groups

Our earlier discussion of the plus-construction showed the importance of the class of acyclic spaces. When these spaces are classifying spaces of discrete groups, as occurs in the context of the Kan–Thurston theorem, the group is also called *acyclic*. Acyclic groups therefore form a useful subclass of the class of all perfect groups. (An acyclic group G is one with $H_i(G) = 0$ for all $i \geq 1$; it must be perfect because $H_1(G) = 0$.) Below is a selection of my favourite examples. (This has been described as the 'zoo', a fair term since a little taxonomy is possible. It was partly exhibited in [11].) The list is more or less in chronological order. It is notable that it is drawn from a very wide range of mathematics. We shall see that there are some interesting, and surprising, patterns. One of these is worth highlighting before we begin.

A group G is called *binate* [11] if it is the direct limit of subgroups G_λ where for each λ there exist $\mu \geq \lambda$, $u_\lambda \in G_\mu - G_\lambda$ and $\phi_\lambda : G_\lambda \to G_\mu$ such that for all $g \in G_\lambda$

$$g = [u_\lambda, \phi_\lambda g]. \tag{3.0.1}$$

Evidently a binate group is perfect; in fact, every element is a commutator. Since any finitely generated subgroup of a binate group G is contained in some G_λ, it excludes u_λ. Thus

- binate groups cannot be finitely generated.

Deeper facts are:

- binate groups are acyclic [11];
- binate groups have no finite-dimensional representations over any field [2];
- every binate group contains infinitely many images of a universal binate group [20].

(Let us agree not to refer to the existence of trivial representations and images.)

3.1 Examples of acyclic groups

3.1.1 Higman's 4-generator, 4-relator group. This is a candidate for being the 'oldest' acyclic group in the literature, although its acyclicity was not proved until much later [32]. It has the presentation

$$\langle x_i \mid x_{i+1} = [x_i, x_{i+1}] \rangle_{i \in \mathbb{Z}/k}$$

where k equals 4 (or any larger integer; $k = 0$ works too, but $k = 1, 2$ or 3 makes the group trivial.) By combinatorial means, Higman [54] shows that it has no finite image and thus, since it is finitely generated, no finite-dimensional representation over any field (for finitely generated subgroups of $\mathrm{GL}_n(F)$ are residually finite). See [21] for generalizations. More geometrically, [32] describes the classifying space of this group as a 2-dimensional complex consisting of one 0-cell, four 1-cells and four 2-cells, and also observed that the group is the fundamental group of a homology 4-sphere. On the other hand, because the group is finitely presented, it must also be the fundamental group of a minimal, symplectic 4-manifold [42]. So it follows from claims of [29] that the free product of two copies of this group is a finitely presented acyclic group that is the fundamental group of a 4-manifold (the connected sum of those above) that has non-trivial Seiberg–Witten invariant yet admits no symplectic structure.

3.1.2 Algebraically closed groups. By analogy with algebraic closure for fields, a (non-trivial) group G is called *algebraically closed* [86] if every finite system of equations consistent with G has a solution in G itself. In particular Equation (3.0.1) always has a solution, and so G is binate. G is also simple [55]. Algebraically closed groups are especially relevant to the study of the Kervaire conjecture cited above [20].

3.1.3 Philip Hall's countable universal locally finite group. This group is built as the union of a nested sequence of finite groups, starting with G_0 of order at least 3. Given G_i, let G_{i+1} be the symmetric group of all permutations of the finite set $G_i \times G_i$, with $G_i \times G_i$ embedded in G_{i+1} by means of the (right) regular representation ρ on the first factor, and G_i in $G_i \times G_i$ by inclusion as the first factor. Then, by the proof of Lemma 1 of [48], any isomorphism between subgroups of $G_i \times G_i$ is expressible as conjugation in G_{i+1}. Let ϕ_i be the embedding of G_i in $G_i \times G_i$ (and thence in G_{i+1}) by inclusion as the second factor. Since this produces a subgroup of $G_i \times G_i$ isomorphic to the one obtained from the diagonal inclusion (the isomorphism is $(\mathrm{id} \times \bar{g}) \mapsto (\bar{g} \times \bar{g})$), we take u_i to be an element of G_{i+1}, conjugation by which restricts to this isomorphism. Then for any g in G_i we have

$$g = [u_i, \phi_i g].$$

It is clearly a countable, locally finite group satisfying P. Hall's condition for universality: that every finite group is embedded, and any two isomorphic finite subgroups are conjugate. Because it is both acyclic and locally finite, this group has the remarkable topological property [16] that there is no homotopically non-trivial map from the classifying space to its 2-skeleton, although of course both these spaces have the same fundamental group. For a generalization to universal locally finite groups made to have the same cardinality as, and to contain, an arbitrary, locally finite group, see [62].

3.1.4 Commutator subgroups of 2-generator, 1-relator groups. This appears to be the first proof of acyclicity of a group in the literature [8]. Let G be a group generated by two elements subject to a single relation. If G_{ab} is infinite cyclic and G' is perfect, then it is shown that G' is acyclic. Remarkably, an example is the geometrically-defined group whose acyclicity was demonstrated just weeks later. In [34], Epstein performs a variant of the grope construction, whereby a doubly-punctured torus is added at each stage. The two punctures bound different loops in the previous stage, one a toral meridian and one a longitude, as prescribed by taking the complement in S^3 of a regular neighbourhood of the 1-complex shown in Fig. 5.

Current work with my colleague Yan-Loi Wong [21], and M. Cencelj in Ljubljana, reveals that the fundamental group of the open aspherical 3-manifold so constructed is after all the commutator subgroup of the group

$$G = \left\langle x, y \mid x = \left[x, yx^{-1}y^{-1}\right] \left[x, y^{-1}xy\right] \right\rangle .$$

Moreover, Higman's group [54] is also of this type, at least when $k = 0$, with the other values of k corresponding to quotient groups.

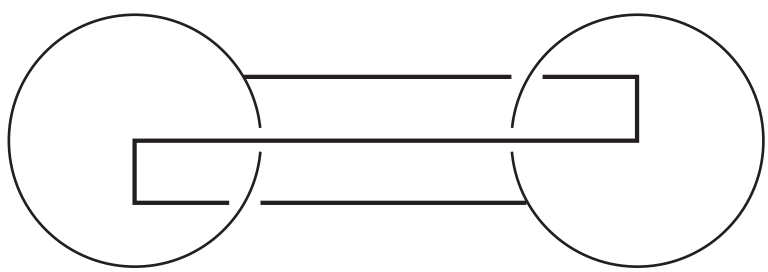

Fig. 5

3.1.5 Stitch groups. In [38], Fox and Artin introduced a wild arc corresponding to iteration (infinitely, towards a cluster point, in both directions) of an underlying pattern called a *stitch* in [21].

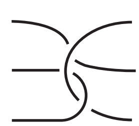

Fig. 6

Stitches also arise in [27], [37], [39], [74], [77]. In general, $\pi_1(S^3 - \kappa)$ is acyclic for any arc, and the commutator subgroup of a finitely presented group for any wild arc κ obtained from a stitch.

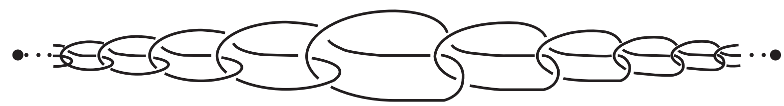

Fig. 7

3.1.6 Groups of self-homeomorphisms with support symmetries. Groups of this kind have been investigated for over six decades, primarily for their conjugacy properties. Many of them are algebraically simple. The first homological treatment was in [69], where the group of self-homeomorphisms of $\mathbb{R}^n$ with compact support was shown to be acyclic (and binate, alias pseudo-mitotic, in [95]). Recall that the support $\text{supp}(h)$ of a self-map h of a topological space is the closure of the set of points x with $h(x) \neq x$. The requirements for a subgroup of a group H of homeomorphisms of a space X to be binate are typically as follows.

Suppose that X has a directed set of subsets $B_\lambda,\ \lambda \in \Lambda$. Put $G = \text{dirlim} G_\lambda$, where $G_\lambda = \{g \in H \mid \text{supp}(g) \subseteq B_\lambda\}$. Suppose that each λ has $\rho_\lambda \in G$, which we refer to as a *dissipator*, that satisfies:

(i) for all $i \geqslant 1,\ \rho_\lambda^i(B_\lambda) \cap B_\lambda = \emptyset$; and

(ii) for all $g \in G_\lambda$ there is a function in H defined by

$$\phi_\lambda(g) = \begin{cases} \rho_\lambda^i g \rho_\lambda^{-i} & \text{on } \rho_\lambda^i(B_\lambda),\ i \geq 1, \\ \text{id} & \text{elsewhere.} \end{cases}$$

A typical picture of a dissipator looks like this:

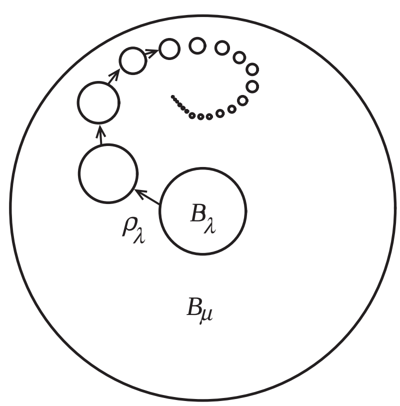

Fig. 8

Given these properties, which may be technically difficult to substantiate (!), then G is a binate group. For by (i), $\text{supp}(\phi_\lambda(g)) \subseteq \text{supp}(\rho_\lambda) \subseteq$ some B_μ. Then by (i) again $\phi_\lambda(G_\lambda)$ commutes with G_λ in G_μ, so that ϕ_λ is a homomorphism. One can check that Equation 3.0.1 holds, with $u_\lambda = \rho_\lambda^{-1}$. So far as I have been able to establish (with the kind assistance of my colleague, Cheng Kai Nah), dissipators first appear in a paper of Schreier and Ulam [85], in the setting of the group of homeomorphisms of the n-ball that fix a neighbourhood of the boundary.

With the addition of a further set-theoretic property, G becomes the minimum normal subgroup of H. Thus when G is H itself, the group G is simple, the case of chief historical interest.

Other previously studied (discrete) groups of homeomorphisms that may retrospectively be shown to be binate by means of dissipators include:

- the (G, K)-rotational examples of [3], namely the Homeo(X) subgroup generated by all homeomorphisms of X that restrict to the identity on some non-empty open subset of X, where X is the rationals, the irrationals, the universal curve, the Cantor ternary set, or, in the case of orientation-preserving homeomorphisms, the universal plane curve or S^n, $n > 1$ (the case $n = 1$ is also claimed in [4], as are some subgroups of the above that leave invariant a given countable dense subset) – see [83] for some proofs, and [84] for some remarkable self-embedding properties;
- the group generated by those homeomorphisms of a topological n-manifold that have support contained in some internal closed n-cell [36];
- for a space X endowed with Euclidean neighbourhoods, the subgroup of Homeo(X) generated by those homeomorphisms that fix a member of a certain family of subsets of X (corresponding to neighbourhoods of the boundary in the n-ball B^n) [99];
- the group of those C^1-diffeomorphisms of a paracompact connected C^1-manifold M that are isotopic to the identity through compactly supported C^1 isotopies [35]. (One might ask whether C^r results also hold, but higher differentiability of $\phi_\lambda(g)$ in (ii) above is difficult to check. Nevertheless, the resulting group is perfect, in fact simple, for $r \neq 1 + \dim M$ [70], [71], [94].)

3.1.7 The general linear group of the cone on a ring. To study the algebraic K-theory of a ring R, [96] embeds R in another ring, its cone CR, whose direct limit general linear group GL(CR) is then shown to be acyclic, containing a copy of GLR as a normal subgroup. In fact, as shown in [11], this group is binate, with dissipator portrayed in [9, p. 85]. Because this dissipator is an infinite permutation matrix, the argument also reveals as binate the group of permutations of $\mathbb{N} \times \mathbb{N}$ that fix the entries of all but finitely many copies of $\mathbb{N}$ (regarded as columns). By means of a bijection of $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$, this shows that the infinite symmetric group $\mathfrak{S}_\infty$, of finitely supported permutations of a countably infinite set, is a normal subgroup of an acyclic group.

3.1.8 The cone on a group. The cone CG on a group G is defined in [61] as the semi-direct product of $G^{\mathbb{Q}}$ by Aut($\mathbb{Q}$). Here $G^{\mathbb{Q}}$ is the set of functions from the rationals $\mathbb{Q}$ to G which map all numbers outside some finite interval to the identity; the group structure on G determines that on $G^{\mathbb{Q}}$. Likewise Aut($\mathbb{Q}$)

denotes the restricted symmetric group on $\mathbb{Q}$ comprising those permutations with compact support. As the reader probably suspects, this group allows dissipators and so is binate. Again, see [11] for details. Observe that G normally embeds in $G^{\mathbb{Q}}$ by means of $g \mapsto b_g$, where $b_g : \mathbb{Q} \to G$ sends 0 to g and non-zero numbers to $1 \in G$. Thus G is embedded as a two-step subnormal subgroup of a binate group.

This raises the interesting issue of which groups can be normal subgroups of acyclic groups. (Some examples occur in (3.1.7) above.) An immediate necessary condition is based on the fact that the image of a perfect group must be perfect. So if N is normal in a perfect group G but the automorphism group of N is hypoabelian, then the map from G to $\mathrm{Aut}(N)$ induced by conjugation must be trivial. In particular, the action of N on itself by conjugation must be trivial, making N abelian.

3.1.9 Diffeomorphism groups for foliation theory. These acyclic groups have significant links to classifying spaces for foliations [94]. Let Y be the interior of a compact manifold $\overline{Y}$ with boundary, and X the complement in $\overline{Y}$ of a closed collar neighbourhood of the boundary of $\overline{Y}$, so that X is an open, relatively compact submanifold of Y diffeomorphic to Y. Then the group of those diffeomorphisms of Y that restrict to the identity on some neighbourhood of the closure of X is acyclic [87]. As noted above, in the attempt to construct dissipators, smoothness of $\phi_\lambda(g)$ is not readily verified. So the argument presented in [87] does not use dissipators explicitly. Likewise, acyclicity of the relative group of volume-preserving diffeomorphisms is treated in [72], [73]. A further variant of this technique is applied to the group of C^1 piecewise $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ homeomorphisms of $\mathbb{R}$ that restrict to the identity near $-\infty$ in [45]. Acyclic groups of orientation-preserving homeomorphisms of $\mathbb{R}$ that are germ-connected to the identity are discussed in [43].

3.1.10 Mitotic groups. These binate groups were first used in [6] to provide combinatorial, finitely generated and, later [7], finitely presented results analogous to those of [61]. Because of their combinatorial construction, they do not have dissipators. They do have the remarkable property that all quotients are also binate.

3.1.11 Perfect, locally free groups. We met such groups in (1.2) above as the fundamental groups of gropes. To deduce acyclicity, we combine the following three facts. Every group is the direct limit of its finitely generated subgroups. Homology preserves direct limits. Free groups have zero homology in all dimensions above the first (for their classifying spaces are bouquets of circles). The conclusion is that a locally free group will be acyclic precisely when it is perfect. The significance of this assertion was emphasized by a construction of Heller [53]: given any element x of a perfect group P, there is a countable, perfect, locally free group D and a homomorphism from D to P whose image contains x. Now, since free products of groups give rise to wedges of classifying spaces, a free product of acyclic groups must be acyclic. It follows that any perfect group is the image of a free product of a suitably large set of representatives of each isomorphism class of countable, perfect, locally free groups (see [16] for related topological subtleties). From one viewpoint, this says that there are enough normal-in-acyclic groups to guarantee the presentation of every perfect group as the image of an acyclic group.

3.1.12 Automorphism groups of large structures. Modelled on (3.1.7) above, a number of automorphism groups are shown, in effect, to have dissipators and so be binate [50]. Examples include the group of all continuous linear automorphisms, or of invertible isometries, of an infinite-dimensional Hilbert space, the group of invertible or of unitary elements in a properly infinite von Neumann algebra, the group of measure-preserving automorphisms of a Lebesgue measure space, and the group of permutations on an arbitrary infinite set. (Note however that the group of finitely supported permutations is not perfect, since the sign representation maps onto the group of order 2.) The class of examples of acyclic groups is greatly enlarged by showing that in many cases the group of unrestricted automorphisms is sufficiently rich in binate subgroups to be acyclic too. It turns out that none of these groups has a finite-dimensional linear representation [14].

3.1.13 Generalized McLain groups. For an arbitrary ring R, the group $M(\Lambda, R)$ of all upper unitriangular matrices over R is acyclic, provided that the ordered set Λ that indexes the entries is dense (in that, between any two elements there lies a third) [10]. Since finitely generated groups of upper unitriangular matrices are nilpotent, $M(\Lambda, R)$ is a locally nilpotent group. If R has a positive characteristic, then $M(\Lambda, R)$ is also locally finite. Moreover, when Λ has both a first and last element, the matrices have a central top-right-hand corner entry, which, together with all entries of the first row may be taken to lie in any right R-module A. This gives a more general group $M(\Lambda, (R, A))$, still acyclic, but having A as its centre. Then the group extension

$$A \hookrightarrow M(\Lambda, (R, A)) \twoheadrightarrow M(\Lambda, (R, A))/A$$

is by Theorem 2.3.6 a universal central extension. In particular, with $R = \mathbb{Z}$, we obtain any abelian group as normal-in-acyclic and as a Schur multiplicator. From our discussion in (3.1.8) above, we see that this construction cannot be generalized to embrace nilpotent groups. For example, the quaternion group of order 8 has hypoabelian automorphism group $\mathfrak{S}_4$, and so is not a normal subgroup of any perfect group.

3.1.14 Torsion-generated acyclic groups. Specific instances of acyclic groups generated by their elements of finite order have been encountered in (3.1.3), (3.1.13) above. It turns out that such torsion-generated acyclic groups share with binate groups the property of lacking finite-dimensional complex representations [12]. This places severe limitations on those groups that can be normal in such an acyclic group. In [18] is constructed a universal finitely presented acyclic group that is strongly torsion generated (normally generated by a single element of arbitrary finite order, hence perfect). (However, the construction does not reveal the number of generators or relations required.) Related results from [18] of interest include the following:

- every sequence of abelian groups is realizable as the higher homology groups of a strongly torsion generated group;
- if a torsion generated group has only finitely many non-zero homology groups, then it is perfect;
- if a locally finite group has only finitely many non-zero homology groups, then it is acyclic (H.-W. Henn).

See also [17] for p-primary refinements of the last two results.

3.1.15 An extension of the braid group. The braid group B_n on n strands can be described geometrically as the group of isotopy classes of orientation-preserving, boundary-fixing homeomorphisms of the disk B^2 that permute a distinguished set X_n of n interior points. Restriction to the permutation of the distinguished points gives an epimorphism from B_n to the symmetric group $\mathfrak{S}_n$. Algebraically, B_n has $n-1$ generators $\sigma_1, ..., \sigma_{n-1}$ and relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad\qquad |i-j|>1. \end{aligned}$$

Then σ_i maps to the transposition $(i \quad i+1)$ to give the epimorphism to $\mathfrak{S}_n$.

Compatibly with the inclusion of X_n in X_{n+1} there is an embedding of $\mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$ and B_n in B_{n+1}. On passing to the direct limit one obtains the group B_∞, which maps onto $\mathfrak{S}_\infty$ and turns out to have the same homology as the double loop-space of S^3. Compatibly with the representation of $\mathfrak{S}_\infty$ as a normal-in-acyclic group in [96] (referred to in (3.1.7) above), there is an acyclic group in which B_∞ is normally embedded. It is constructed in [44] as a group of automorphisms of a generalized braid group for which the above set X_n is replaced by a tree whose vertices are indexed by the dyadic numbers between 0 and 1.

3.1.16 The universal binate tower. Although we have seen that it commonly occurs in the geometric setting of dissipators, the characteristic equation (3.0.1) for binate groups may be viewed as a purely algebraic phenomenon. This suggests a universal example [11]. Starting with any group H as base (the trivial group gives the extreme case), one can construct a binate tower by means of HNN-extensions. Let $H_0 = H$ and for each $i \geq 0$

$$H_{i+1} = \mathrm{gp}\left\langle H_i \times H_i, u_i \mid (g,g) = u_i(1,g)u_i^{-1} \text{ for each } g \in H_i \right\rangle.$$

With H_i embedded in H_{i+1} as $H_i \times 1$ and $\phi_i(g) = (1,g)$, the binate equation is readily checked, making the direct limit a binate group (even though the intermediate groups have hyperexponentially-growing homology). This tower is the initial object in a category of binate towers with base group H [20].

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