1

THE SURGERY CLASSIFICATION OF MANIFOLDS

Chapter 1 is an introduction to the surgery method of classifying manifolds.

Manifolds are understood to be differentiable, compact and closed, unless otherwise specified.

A classification of manifolds up to diffeomorphism requires the construction of a complete set of algebraic invariants such that:

(i) the invariants of a manifold are computable,
(ii) two manifolds are diffeomorphic if and only if they have the same invariants,
(iii) there is given a list of non-diffeomorphic manifolds realizing every possible set of invariants.

One could also seek a homotopy classification of manifolds, asking for a complete set of invariants for distinguishing the homotopy types of manifolds. Diffeomorphic manifolds are homotopy equivalent.

The most important invariant of a manifold M^m is its dimension, the number $m \geqslant 0$ such that M is locally diffeomorphic to the Euclidean space $\mathbb{R}^m$. If $m \neq n$ then $\mathbb{R}^m$ is not diffeomorphic to $\mathbb{R}^n$, so that an m-dimensional manifold M^m cannot be diffeomorphic to an n-dimensional manifold N^n. The homology and cohomology of an orientable m-dimensional manifold M are related by the Poincaré duality isomorphisms

$$H^*(M) \cong H_{m-*}(M).$$

Any m-dimensional manifold M has $\mathbb{Z}_2$-coefficient Poincaré duality

$$H^*(M;\mathbb{Z}_2) \cong H_{m-*}(M;\mathbb{Z}_2),$$

with

$$H_m(M;\mathbb{Z}_2) = \mathbb{Z}_2, H_n(M;\mathbb{Z}_2) = 0 \text{ for } n > m.$$

The dimension of a manifold M is thus characterized homologically as the largest integer $m \geqslant 0$ with $H_m(M;\mathbb{Z}_2) \neq 0$. Homology is homotopy invariant, so that the dimension is also a homotopy invariant: if $m \neq n$ an m-dimensional manifold M^m cannot be homotopy equivalent to an n-dimensional manifold N^n.

There is a complete diffeomorphism classification of m-dimensional manifolds only in the dimensions $m = 0, 1, 2$, where it coincides with the homotopy classification. For $m \geqslant 3$ there exist m-dimensional manifolds which are homotopy equivalent but not diffeomorphic, so that the diffeomorphism and homotopy classifications must necessarily differ. For $m = 3$ complete classifications are theoretically possible, but have not been achieved in practice—the Poincaré conjecture that every 3-dimensional manifold homotopy equivalent to S^3 is actually diffeomorphic to S^3 remains unsolved!

For $m \geqslant 4$ group-theoretic decision problems prevent a complete classification of m-dimensional manifolds, by the following argument. Every manifold M can be triangulated by a finite simplicial complex, so that the fundamental group $\pi_1(M)$ is finitely presented. Homotopy equivalent manifolds have isomorphic fundamental groups. Every finitely presented group arises as the fundamental group $\pi_1(M)$ of an m-dimensional manifold M. It is not possible to have a complete set of invariants for distinguishing the isomorphism class of a group from a finite presentation. Group-theoretic considerations thus make the following questions unanswerable in general:

(a) *Is M homotopy equivalent to M'?*
(b) *Is M diffeomorphic to M'?*

since already the question

(c) *Is $\pi_1(M)$ isomorphic to $\pi_1(M')$?*

is unanswerable in general.

The surgery method of classifying manifolds seeks to answer a different question:
Given a homotopy equivalence of m-dimensional manifolds $f : M \to M'$ is f homotopic to a diffeomorphism?

Every homotopy equivalence of 2-dimensional manifolds (= surfaces) is homotopic to a diffeomorphism, by the nineteenth century classification of surfaces which is recalled in Chapter 3.

A homotopy equivalence of 3-dimensional manifolds is not in general homotopic to a diffeomorphism. The first examples of such homotopy equivalences appeared in the classification of the 3-dimensional lens spaces in the 1930s: the Reidemeister torsion of a lens space is a diffeomorphism invariant which is not homotopy invariant. Algebraic K-theory invariants such as Reidemeister and Whitehead torsion are significant in the classification of manifolds with finite fundamental group, and in deciding if 'h-cobordant' manifolds are diffeomorphic (via the s-Cobordism Theorem, stated in 1.11 below), but they are too special to decide if an arbitrary homotopy equivalence of manifolds is homotopic to a diffeomorphism. Chapter 8 deals with the main applications of Whitehead torsion to the topology of manifolds.

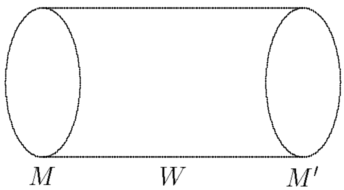
In 1956, Milnor [49] constructed an exotic sphere, a differentiable manifold Σ^7 with a homotopy equivalence (in fact a homeomorphism) $\Sigma^7 \to S^7$ which is not homotopic to a diffeomorphism. The subsequent classification by Kervaire and Milnor [38] for $m \geqslant 5$ of pairs

$$(m\text{-dimensional manifold } \Sigma^m, \text{ homotopy equivalence } \Sigma^m \to S^m\ )$$

was the first triumph of surgery theory. It remains the best introduction to surgery, particularly as it deals with simply-connected manifolds M (i.e. those with $\pi_1(M) = \{1\}$) and so avoids the fundamental group. The surgery classification of homotopy spheres is outlined in Section 13.3.

Definition 1.1 An $(m+1)$-*dimensional cobordism* $(W; M, M')$ is an $(m+1)$-dimensional manifold W^{m+1} with boundary the disjoint union of closed m-dimensional manifolds M, M'

$$\partial W = M \cup M'.$$



□

The cobordism classes of manifolds are groups, with addition by disjoint union. The computation of the cobordism groups was a major achievement of topology in the 1950s—Chapter 6 is an introduction to cobordism theory. The cobordism classification of manifolds is very crude: for example, the 0- and 2-dimensional cobordism groups have order two, and the 1- and 3-dimensional cobordism groups are trivial. Surgery theory applies the methods of cobordism theory to the rather more delicate classification of the homotopy types of manifolds.

What is surgery?

Definition 1.2 A *surgery* on an m-dimensional manifold M^m is the procedure of constructing a new m-dimensional manifold

$$M'^m = \text{cl.}(M \backslash S^n \times D^{m-n}) \cup_{S^n \times S^{m-n-1}} D^{n+1} \times S^{m-n-1}$$

by cutting out $S^n \times D^{m-n} \subset M$ and replacing it by $D^{n+1} \times S^{m-n-1}$. The surgery *removes* $S^n \times D^{m-n} \subset M$ and *kills* the homotopy class $S^n \to M$ in $\pi_n(M)$. □

Terminology: given a subset $Y \subseteq X$ of a space X write cl.(Y) for the *closure* of Y in X, the intersection of all the closed subsets $Z \subseteq X$ with $Y \subseteq Z$.

At first sight, it might seem surprising that surgery can be used to answer such a delicate question as whether a homotopy equivalence of manifolds is homotopic to a diffeomorphism, since an individual surgery has such a drastic effect on the homotopy type of a manifold:

Example 1.3

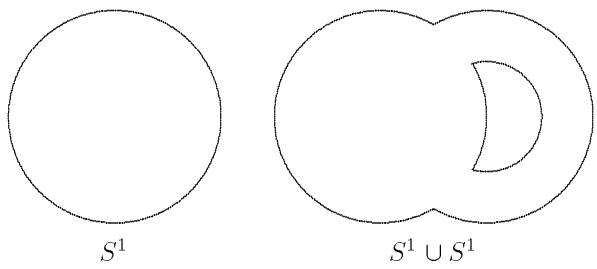
(i) View the m-sphere S^m as

$$S^m = \partial(D^{n+1} \times D^{m-n}) = S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1}.$$

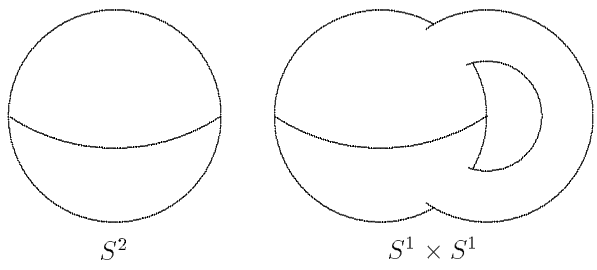
The surgery on S^m removing $S^n \times D^{m-n} \subset S^m$ converts the m-sphere S^m into the product of spheres

$$D^{n+1} \times S^{m-n-1} \cup D^{n+1} \times S^{m-n-1} = S^{n+1} \times S^{m-n-1}.$$

(ii) For $m = 1$, $n = 0$ the surgery of (i) converts the circle S^1 into the disjoint union $S^0 \times S^1 = S^1 \cup S^1$ of two circles.



(iii) For $m = 2$, $n = 0$ the surgery of (i) converts the 2-sphere S^2 into the torus $S^1 \times S^1$.

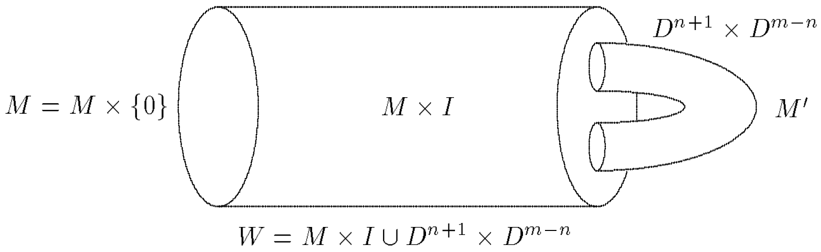


(iv) For $m = n$ the surgery of (i) converts the m-sphere S^m into the empty set $\emptyset$. □

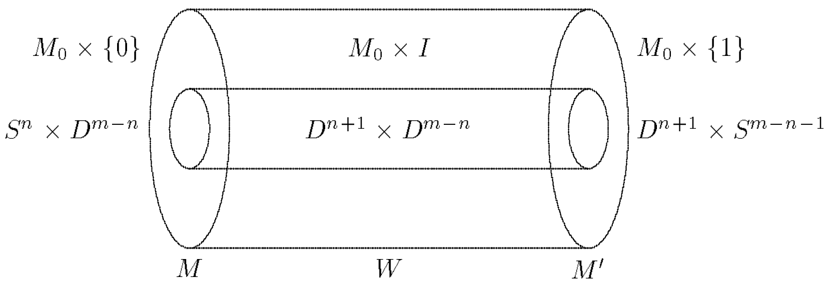
There is an intimate connection between surgery and cobordism. A surgery on a manifold M determines a cobordism $(W; M, M')$:

Definition 1.4 The *trace* of the surgery removing $S^n \times D^{m-n} \subset M^m$ is the cobordism $(W; M, M')$ obtained by attaching $D^{n+1} \times D^{m-n}$ to $M \times I$ at

$$S^n \times D^{m-n} \times \{1\} \subset M \times \{1\}. \qquad \square$$



Here is a more symmetric picture of the trace $(W; M, M')$:



The m-dimensional manifold with boundary

$$(M_0, \partial M_0) = (\mathrm{cl.}(M^m \backslash S^n \times D^{m-n}), S^n \times S^{m-n-1})$$

is obtained from M by cutting out the interior of $S^n \times D^{m-n} \subset M$, with

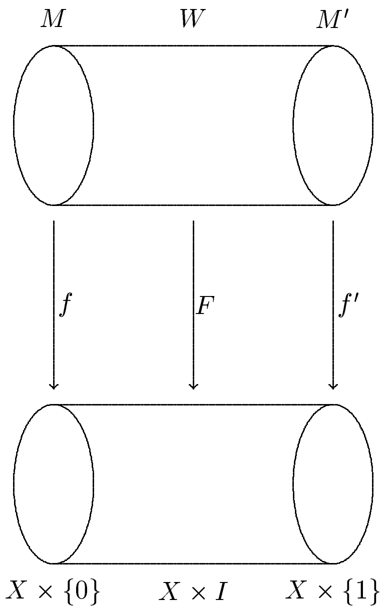
$$\begin{aligned}
&M = M_0 \cup_{\partial M_0} S^n \times D^{m-n},\\
&M' = M_0 \cup_{\partial M_0} D^{n+1} \times S^{m-n-1},\\
&W = (M_0 \times I) \cup (D^{n+1} \times D^{m-n}),\\
&(M_0 \times I) \cap (D^{n+1} \times D^{m-n}) = S^n \times S^{m-n-1} \times I.
\end{aligned}$$

Note that M is obtained from M' by the opposite surgery removing $D^{n+1} \times S^{m-n-1} \subset M'$.

In fact, two m-dimensional manifolds M^m, M'^m are cobordant if and only if M' can be obtained from M by a finite sequence of surgeries.

Definition 1.5 A *bordism* of maps $f : M^m \to X$, $f' : M'^m \to X$ from m-dimensional manifolds to a space X is a cobordism $(W; M, M')$ together with a map

$$(F; f, f') : (W; M, M') \to X \times (I; \{0\}, \{1\}).$$



□

Example 1.6 A homotopy $h : f \simeq f' : M^m \to X$ can be regarded as a bordism

$$(F; f, f') : (W; M, M') = M \times (I; \{0\}, \{1\}) \to X \times (I; \{0\}, \{1\}),$$

with

$$F : M \times I \to X \times I ; (x, t) \mapsto (h(x, t), t).$$ □

Diffeomorphic manifolds are cobordant. Homotopy equivalent closed manifolds are cobordant, but in general only by a nonorientable cobordism. It is possible to decide if two manifolds are cobordant, but it is not possible to decide if cobordant manifolds are homotopy equivalent, or if homotopy equivalent manifolds are diffeomorphic. Given that cobordism is considerably weaker than diffeomorphism and that cobordism drastically alters homotopy types, it may appear surprising that cobordism is a sufficiently powerful tool to distinguish manifolds within a homotopy type. However, surgery theory provides a systematic procedure for deciding if a map of m-dimensional manifolds $f : M \to M'$ satisfying certain bundle-theoretic conditions is bordant to a homotopy equivalence, and if the bordism can be chosen to be a homotopy (as in 1.6), at least in dimensions $m \geqslant 5$. This works because surgery makes it comparatively easy to construct cobordisms with prescribed homotopy types. The applications of cobordism theory to the surgery classification of high-dimensional manifolds depend on the following fundamental result:

Whitney Embedding Theorem 1.7 ([97], [99], 1944)
If $f : N^n \to M^m$ is a map of manifolds such that

$$\text{either} \quad 2n+1 \leqslant m$$
$$\text{or} \quad m = 2n \geqslant 6 \text{ and } \pi_1(M) = \{1\}$$

then f is homotopic to an embedding $N^n \hookrightarrow M^m$. □

The proof of 1.7 will be outlined in Chapter 7.

Definition 1.8 (i) An *h-cobordism* is a cobordism $(W^{m+1}; M^m, M'^m)$ such that the inclusions $M \hookrightarrow W$, $M' \hookrightarrow W$ are homotopy equivalences.
(ii) An h-cobordism $(W; M, M')$ is *trivial* if there exists a diffeomorphism

$$(W; M, M') \cong M \times (I; \{0\}, \{1\})$$

which is the identity on M, in which case the composite homotopy equivalence $M \simeq W \simeq M'$ is homotopic to a diffeomorphism. □

The h-Cobordism Theorem was the crucial first step in the homotopy classification of high-dimensional manifolds:

***h*-Cobordism Theorem 1.9** (Smale [83], 1962)
A simply-connected $(m+1)$-dimensional h-cobordism $(W^{m+1}; M, M')$ with $m \geqslant 5$ is trivial. □

Thus for $m \geqslant 5$ simply-connected m-dimensional manifolds M, M' are diffeomorphic if and only if they are h-cobordant.

The h-Cobordism Theorem was subsequently generalized to non-simply-connected manifolds, using Whitehead torsion (which is described in Chapter 8). The Whitehead group $Wh(\pi)$ of a group π is an abelian group which measures the extent to which Gaussian elimination fails for invertible matrices with entries in the group ring $\mathbb{Z}[\pi]$. The Whitehead torsion of a homotopy equivalence $f : M^m \to M'^m$ of manifolds (or more generally of finite CW complexes) is an element $\tau(f) \in Wh(\pi_1(M))$. A homotopy equivalence f is *simple* if $\tau(f) = 0$.

Definition 1.10 An *s-cobordism* is a cobordism $(W^{m+1}; M^m, M'^m)$ such that the inclusions $M \hookrightarrow W$, $M' \hookrightarrow W$ are simple homotopy equivalences. □

A diffeomorphism $f : M^m \to M'^m$ of m-dimensional manifolds determines an $(m+1)$-dimensional s-cobordism $(W; M, M')$ with

$$W = (M \times I \cup M')/\{(x,1) \sim f(x) | x \in M\}$$

the mapping cylinder, such that there is defined a diffeomorphism

$$(W; M, M') \cong M \times (I; \{0\}, \{1\}).$$

The s-Cobordism Theorem is the non-simply-connected version of the h-Cobordism Theorem:

s-Cobordism Theorem 1.11 (Barden-Mazur-Stallings 1964)
An $(m+1)$-dimensional h-cobordism $(W^{m+1}; M, M')$ with $m \geqslant 5$ is trivial if and only if it is an s-cobordism. $\square$

It follows that for $m \geqslant 5$ h-cobordant m-dimensional manifolds M, M' are diffeomorphic if and only if they are s-cobordant. The proofs of the h- and s-Cobordism Theorems will be outlined in Chapter 8.

The Whitehead group of the trivial group is trivial, $Wh(\{1\}) = 0$, and the h-Cobordism Theorem is just the simply-connected special case of the s-Cobordism Theorem. The condition $m \geqslant 5$ in the h- and s-Cobordism Theorems is due to the use of the Whitney Embedding Theorem (1.7) in their proof. It is known that the h- and s-Cobordism Theorems for $(m+1)$-dimensional cobordisms are true for $m = 0, 1$, and are false for $m = 4$ (Donaldson [21]), $m = 3$ (Cappell and Shaneson [18]). It is not known if they are true for $m = 2$, on account of the classical 3-dimensional Poincaré conjecture.

Milnor [58] used the lens spaces to construct h-cobordisms $(W^{m+1}; M, M')$ of non-simply-connected manifolds which are not diffeomorphic.

One way to prove that manifolds are diffeomorphic is to first decide if they are cobordant, and then to decide if some cobordism can be modified by successive surgeries on the interior to be an s-cobordism.

The tangent bundle of an m-dimensional manifold M^m is classified by the homotopy class of a map

$$\tau_M : M \to BO(m).$$

(See Chapter 5 for some basic information on bundles, including the classifying space $BO(m)$.) If $f : M \to M'$ is a homotopy equivalence of m-dimensional manifolds which is homotopic to a diffeomorphism there exists a homotopy $f^*\tau_{M'} \simeq \tau_M : M \to BO(m)$. The tubular neighbourhood of an embedding $M^m \hookrightarrow S^{m+k}$ (k large) is a k-plane bundle $\nu_M : M \to BO(k)$ which is a stable inverse of τ_M. The stable normal bundle of M is classified by a map

$$\nu_M : M \to BO = \varinjlim_k BO(k).$$

By the result of Mazur [47] for $m \geqslant 5$ a homotopy equivalence of m-dimensional manifolds $f : M \to M'$ is covered by a stable bundle map $b : \nu_M \to \nu_{M'}$ if and only if $f \times 1 : M \times \mathbb{R}^k \to M' \times \mathbb{R}^k$ is homotopic to a diffeomorphism for some $k \geqslant 0$. It is possible to extend f to a homotopy equivalence of h-cobordisms

$$(F; f, 1) : (W; M, M') \to M' \times (I; \{0\}, \{1\})$$

if and only if $f \times 1 : M \times \mathbb{R} \to M' \times \mathbb{R}$ is homotopic to a diffeomorphism.

The surgery theory developed by Browder, Novikov, Sullivan, and Wall in the 1960s provides a systematic solution to the problem of deciding if a homotopy equivalence $f : M \to M'$ of m-dimensional manifolds is homotopic to a diffeomorphism, with obstructions taking values in the topological K-theory of vector bundles and the algebraic L-theory of quadratic forms. The obstruction theory was obtained as the relative version of the systematic solution to the problem of deciding if a space X with m-dimensional Poincaré duality $H^*(X) \cong H_{m-*}(X)$ is homotopy equivalent to an m-dimensional manifold. The theory thus deals both with the existence and the uniqueness of manifold structures in homotopy types.

Definition 1.12 An *m-dimensional geometric Poincaré complex* is a finite *CW* complex X with a fundamental homology class $[X] \in H_m(X)$ (using twisted coefficients in the nonorientable case) such that the cap products are isomorphisms

$$[X] \cap - : H^*(X; \Lambda) \to H_{m-*}(X; \Lambda)$$

for every $\mathbb{Z}[\pi_1(X)]$-module Λ. □

Example 1.13 An m-dimensional manifold is an m-dimensional geometric Poincaré complex. □

The property of being a geometric Poincaré complex is homotopy invariant, unlike the property of being a manifold. Thus any finite *CW* complex homotopy equivalent to a manifold is a geometric Poincaré complex. In order for a space to have a fighting chance of being homotopy equivalent to an m-dimensional manifold it must at least be homotopy equivalent to an m-dimensional geometric Poincaré complex. Geometric Poincaré complexes which are not homotopy equivalent to a manifold may be obtained by glueing together m-dimensional manifolds with boundary $(M, \partial M)$, $(M', \partial M')$ using a homotopy equivalence $\partial M \simeq \partial M'$ which is not homotopic to a diffeomorphism.

Definition 1.14 Let X be an m-dimensional geometric Poincaré complex.

(i) A *manifold structure* (M, f) on X is an m-dimensional manifold M together with a homotopy equivalence $f : M \to X$.

(ii) The *manifold structure set* $\mathscr{S}(X)$ of X is the set of equivalence classes of manifold structures (M, f), subject to the equivalence relation:

$$\begin{aligned}&(M, f) \sim (M', f') \quad \text{if there exists a bordism} \\ &\quad (F; f, f') : (W; M, M') \to X \times (I; \{0\}, \{1\}) \text{ with } F \\ &\quad \text{a homotopy equivalence, so that } (W; M, M') \text{ is an } h\text{-cobordism.}\end{aligned}$$

□

Surgery theory asks: is $\mathscr{S}(X)$ non-empty? And if so, then how large is it? In any case, it is clear from the definition that $\mathscr{S}(X)$ is a homotopy invariant of X, that is, that a homotopy equivalence $X \to Y$ induces a bijection $\mathscr{S}(X) \to \mathscr{S}(Y)$. Surgery theory reduces $\mathscr{S}(X)$ to more familiar homotopy invariant objects associated to X. A homotopy equivalence $f : M^m \to N^m$ of m-dimensional manifolds determines an element $(M, f) \in \mathscr{S}(N)$, such that f is h-cobordant to $1 : N \to N$ if and only if

$$(M, f) = (N, 1) \in \mathscr{S}(N).$$

In particular, if f is homotopic to a diffeomorphism then f is h-cobordant to $1 : N \to N$, and $(M, f) = (N, 1) \in \mathscr{S}(N)$.

The determination of $\mathscr{S}(X)$ is closely related to the bundle properties of manifolds and geometric Poincaré complexes.

A finite CW complex X is an m-dimensional geometric Poincaré complex if and only if a regular neighbourhood $(Y, \partial Y) \subset S^{m+k}$ of an embedding $X \hookrightarrow S^{m+k}$ is such that

$$\text{mapping fibre } (\partial Y \to Y) \simeq S^{k-1}.$$

A regular neighbourhood is the PL analogue of a tubular neighbourhood.

The $(k-1)$-spherical fibration

$$S^{k-1} \to \partial Y \to Y \simeq X$$

is the *Spivak normal fibration* of a geometric Poincaré complex X, with a classifying map

$$\nu_X : X \to BG = \varinjlim_k BG(k).$$

(See Section 9.2 for an exposition of fibrations.) The Spivak normal fibration is the homotopy theoretic analogue of the stable normal bundle $\nu_M = -\tau_M$ of a manifold M.

The classifying spaces BO, BG for stable bundles and spherical fibrations are related by a fibration sequence

$$G/O \to BO \to BG \to B(G/O),$$

with G/O the classifying space for stable bundles with a fibre homotopy trivialization. The homotopy class of the composite map

$$t(\nu_X) : X \xrightarrow{\nu_X} BG \longrightarrow B(G/O)$$

is the primary obstruction to X being homotopy equivalent to an m-dimensional manifold. There exists a null-homotopy $t(\nu_X) \simeq \{*\}$ if and only if the Spivak normal fibration ν_X admits a vector bundle reduction $\tilde{\nu}_X : X \to BO$. Surgery theory offers a two-stage programme for deciding if a geometric Poincaré complex

X is homotopy equivalent to a manifold, involving the concept of a normal map:

Definition 1.15 A *degree 1 normal map* from an m-dimensional manifold M^m to an m-dimensional geometric Poincaré complex X

$$(f, b) : M^m \to X$$

is a map $f : M \to X$ such that

$$f_*[M] = [X] \in H_m(X),$$

together with a stable bundle map $b : \nu_M \to \eta$ over f, from the stable normal bundle $\nu_M : M \to BO$ to a stable bundle $\eta : X \to BO$. □

The two stages of the obstruction theory for deciding if an m-dimensional geometric Poincaré complex X is homotopy equivalent to an m-dimensional manifold are:

(i) Does X admit a degree 1 normal map $(f, b) : M^m \to X$? This is the case precisely when the map $t(\nu_X) : X \to B(G/O)$ is null-homotopic.
(ii) If the answer to (i) is yes, is there a degree 1 normal map $(f, b) : M^m \to X$ which is bordant to a homotopy equivalence $(f', b') : M'^m \to X$?

The extent to which a degree 1 normal map $(f, b) : M^m \to X$ of connected M, X fails to be a homotopy equivalence is measured by the relative homotopy groups $\pi_{n+1}(f)$ $(n \geqslant 0)$ of pairs of elements

$$(\text{map } g : S^n \to M, \text{ null-homotopy } h : fg \simeq * : S^n \to X).$$

By Whitehead's Theorem, f is a homotopy equivalence if and only if $\pi_*(f) = 0$. Let $m = 2n$ or $2n + 1$. It turns out that it is always possible to 'kill' $\pi_i(f)$ for $i \leqslant n$, meaning that there is a bordant degree 1 normal map $(f', b') : M' \to X$ with $\pi_i(f') = 0$ for $i \leqslant n$. There exists a normal bordism of (f, b) to a homotopy equivalence if and only if it is also possible kill $\pi_{n+1}(f')$. In general there is an obstruction to killing $\pi_{n+1}(f')$, which for $m \geqslant 5$ is essentially algebraic in nature:

Wall Surgery Obstruction Theorem 1.16 ([92], 1970)
For any group π there are defined algebraic L-groups $L_m(\mathbb{Z}[\pi])$ depending only on $m(\text{mod} 4)$, as groups of stable isomorphism classes of $(-1)^n$-quadratic forms over $\mathbb{Z}[\pi]$ for $m = 2n$, and as groups of stable automorphisms of such forms for $m = 2n + 1$. An m-dimensional degree 1 normal map $(f, b) : M^m \to X$ has a **surgery obstruction**

$$\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)]),$$

such that $\sigma_*(f, b) = 0$ if (and for $m \geqslant 5$ only if) (f, b) is bordant to a homotopy equivalence. □

If $m = 2n \geqslant 6$ and $(f, b) : M^{2n} \to X$ is a degree 1 normal map such that $\pi_i(f) = 0$ for $i \leqslant n$ the surgery obstruction is largely determined by the $(-1)^n$-symmetric pairing

$$\lambda : K \times K \to \mathbb{Z}[\pi_1(X)]$$

defined on the kernel $\mathbb{Z}[\pi_1(X)]$-module $K = \pi_{n+1}(f)$ by the intersections of immersions $S^n \looparrowright M^{2n}$ which are null-homotopic in X. In order to kill K it is necessary that there be a sufficient number of elements $x \in K$ with $\lambda(x, x) = 0$ which are represented by embeddings $x : S^n \times D^n \hookrightarrow M^{2n}$. The even-dimensional surgery obstruction will be obtained in Chapter 11, and involves a $(-1)^n$-quadratic refinement μ of the $(-1)^n$-symmetric form (K, λ). The odd-dimensional surgery obstruction for $m = 2n + 1 \geqslant 5$ will be obtained in Chapter 12.

Example 1.17 The simply-connected surgery obstruction groups are given by:

m (mod4)	0	1	2	3
$L_m(\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0

The surgery obstruction of a $4k$-dimensional normal map $(f, b) : M^{4k} \to X$ with $\pi_1(X) = \{1\}$ is

$$\sigma_*(f, b) = \tfrac{1}{8}\text{signature } (K_{2k}(M), \lambda) \in L_{4k}(\mathbb{Z}) = \mathbb{Z}$$

with λ the nonsingular symmetric form on the middle-dimensional homology kernel $\mathbb{Z}$-module

$$K_{2k}(M) = \ker(f_* : H_{2k}(M) \to H_{2k}(X)).$$

The surgery obstruction of a $(4k+2)$-dimensional normal map $(f, b) : M^{4k+2} \to X$ with $\pi_1(X) = \{1\}$ is

$$\sigma_*(f, b) = \text{Arf invariant } (K_{2k+1}(M; \mathbb{Z}_2), \lambda, \mu) \in L_{4k+2}(\mathbb{Z}) = \mathbb{Z}_2$$

with λ, μ the nonsingular quadratic form on the middle-dimensional $\mathbb{Z}_2$-coefficient homology kernel $\mathbb{Z}_2$-module

$$K_{2k+1}(M; \mathbb{Z}_2) = \ker(f_* : H_{2k+1}(M; \mathbb{Z}_2) \to H_{2k+1}(X; \mathbb{Z}_2)).$$

□

Surgery Exact Sequence 1.18 (Browder, Novikov, Sullivan, Wall 1962–1970) *Let $m \geqslant 5$.*

(i) *The manifold structure set $\mathscr{S}(X)$ of an m-dimensional geometric Poincaré complex X is non-empty if and only if there exists a normal map $(f, b) : M^m \to X$ with surgery obstruction*

$$\sigma_*(f, b) = 0 \in L_m(\mathbb{Z}[\pi_1(X)]).$$

(ii) *The structure set* $\mathscr{S}(M)$ *of an* m*-dimensional manifold* M^m *fits into the* **surgery exact sequence** *of pointed sets*

$$\cdots \to L_{m+1}(\mathbb{Z}[\pi_1(M)]) \to \mathscr{S}(M) \to [M, G/O] \to L_m(\mathbb{Z}[\pi_1(M)]). \qquad \square$$

The surgery exact sequence will be obtained in Chapter 13. The restriction $m \geqslant 5$ is due to the use of the Whitney Embedding Theorem (1.7) in the proof, exactly as in the h- and s-Cobordism Theorems.

The geometric surgery construction works just as well in the low dimensions, $m \leqslant 4$. However, the possible geometric surgeries and their effect (e.g. on the fundamental group) are much harder to relate to algebra than in the higher dimensions. The type of algebraic surgery considered in the book thus only deals with the 'high-dimensional' part of 3- and 4-dimensional topology.