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Introduction

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1.1 Overview

Quantum field theory is a synthesis of quantum mechanics and special relativity, and it is one of the great achievements of modern physics. Quantum mechanics, as formulated by Bohr, Heisenberg, Schrödinger, Pauli, Dirac, and many others, is an intrinsically non-relativistic theory. To make it consistent with special relativity, the real problem is not to find a relativistic generalization of the Schrödinger equation.[1] Wave equations, relativistic or not, cannot account for processes in which the number and the type of particles changes, as in almost all reactions of nuclear and particle physics. Even the process of an atomic transition from an excited atomic state A^* to a state A with emission of a photon, $A^* \to A + \gamma$, is in principle unaccessible to this treatment (although in this case, describing the electromagnetic field classically and the atom quantum mechanically, one can get some correct results, even if in a not very convincing manner). Furthermore, relativistic wave equations suffer from a number of pathologies, like negative-energy solutions.

A proper resolution of these difficulties implies a change of viewpoint, from wave equations, where one quantizes a single particle in an external classical potential, to quantum field theory, where one identifies the particles with the modes of a field, and quantizes the field itself. The procedure also goes under the name of second quantization.

[1] Actually, Schrödinger first found a relativistic equation, that today we call the Klein–Gordon equation. He then discarded it because it gave the wrong fine structure for the hydrogen atom, and he retained only the non-relativistic limit. See Weinberg (1995), page 4.

The methods of quantum field theory (QFT) have great generality and flexibility and are not restricted to the domain of particle physics. In a sense, field theory is a universal language, and it permeates many branches of modern research. In general, field theory is the correct language whenever we face collective phenomena, involving a large number of degrees of freedom, and this is the underlying reason for its unifying power. For example, in condensed matter the excitations in a solid are quanta of fields, and can be studied with field theoretical methods. An especially interesting example of the unifying power of QFT is given by the phenomenon of superconductivity which, expressed in the field theory language, turns out to be conceptually the same as the Higgs mechanism in particle physics. As another example we can mention that the Feynman path integral, which is a basic tool of modern quantum field theory, provides a formal analogy between field theory and statistical mechanics, which has stimulated very important exchanges between these two areas. Beside playing a crucial role for physicists, quantum field theory even plays a role in pure mathematics, and in the last 20 years the physicists' intuition stemming in particular from the path integral formulation of QFT has been at the basis of striking and unexpected advances in pure mathematics.

QFT obtains its most spectacular successes when the interaction is small and can be treated perturbatively. In quantum electrodynamics (QED) the theory can be treated order by order in the *fine structure constant* $\alpha = e^2/(4\pi\hbar c) \simeq 1/137$. Given the smallness of this parameter, a perturbative treatment is adequate in almost all situations, and the agreement between theoretical predictions and experiments can be truly spectacular. For example, the electron has a magnetic moment of modulus $g|e|\hbar/(4m_e c)$, where g is called the gyromagnetic ratio. While classical electrodynamics erroneously suggests $g = 1$, the Dirac equation gives $g = 2$, and QED predicts a small deviation from this value; the experimentally measured value is

$$\left(\frac{g-2}{2}\right)\Big|_{\rm exp} = 0.001\ 159\ 652\ 187(4) \tag{1.1}$$

(the digit in parentheses is the experimental error on the last figure), and the theoretical prediction, computed perturbatively up to order α^4, is

$$\left(\frac{g-2}{2}\right)\Big|_{\rm th} = \frac{\alpha}{2\pi} - (0.328\,478\,965\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.176\,11\ldots)\left(\frac{\alpha}{\pi}\right)^3$$
$$-(1.434\ldots)\left(\frac{\alpha}{\pi}\right)^4 = 0.001\ 159\ 652\ 140(5)(4)(27)\,.$$

Different sources of errors on the last figures are written separately in parentheses. The theoretical error is due partly to the numerical evaluation of Feynman diagrams (there are 891 of them at order α^4!) and partly to the fact that, at this level of precision, hadronic contributions come into play. We also need to know α with sufficient accuracy; this is provided by the quantum Hall effect.

The gyromagnetic ratio has been measured very precisely also for the muon, and the accuracy of this measurement has been improved recently,[2] with the result $(g-2)/2|_{\rm exp} = 0.001\,165\,9208(6)$, and a theoretical prediction $(g-2)/2|_{\rm th} = 0.001\,165\,9181(7)$. The remaining discrepancy has aroused much interest, in the hope that it might be a signal of new physical effects, but to see whether this is actually the case requires first a better theoretical understanding of hadronic contributions, which are more difficult to compute. In any case, an agreement between theory and experiment at the level of 10 decimal figures for the electron (or eight for the muon) is spectacular, and it is among the most precise in physics.

[2]See http://www.g-2.bnl.gov/. This values updates the value reported in the 2004 edition of the Review of Particle Physics.

As we know today, QED is only a part of a larger theory. As we approach the scales of nuclear physics, i.e. length scales $r \sim 10^{-13}$ cm or energies $E \sim 200$ MeV, the existence of new interactions becomes evident: strong interactions are responsible for instance for binding together neutrons and protons into nuclei, and weak interactions are responsible for a number of decays, like the beta decay of the neutron into the proton, electron and antineutrino, $n \to pe^-\bar{\nu}_e$. A successful theory of beta decay was already proposed by Fermi in 1934. We now understand the Fermi theory as a low energy approximation to a more complete theory, that unifies the weak and electromagnetic interactions into a single conceptual framework, the electroweak theory. This theory, developed in the early 1970s, together with the fundamental theory of strong interactions, quantum chromodynamics (QCD), has such spectacular experimental successes that it now goes under the name of the Standard Model. In the last decade of the 20th century the LEP machine at CERN performed a large number of precision measurements, at the level of one part in 10^4, which are all completely reproduced by the theoretical predictions of the Standard Model. These results show that we do understand the laws of Nature down to the scale of 10^{-17} cm, i.e. four orders of magnitude below the size of a nucleus and nine orders of magnitude below the size of an atom. Part of the activity of high energy physicists nowadays is devoted to the search of physics beyond the Standard Model. The best hint for new physics presently comes from the recent experimental evidence for neutrino oscillations. These oscillations imply that neutrinos have a very small mass, whose deeper origin is suspected to be related to physics beyond the Standard Model.

The Standard Model has a beautiful theoretical structure; its discovery and development, due among others to Glashow, Weinberg, Salam and 't Hooft, requires a number of new concepts compared to QED. A detailed explanation of the Standard Model is beyond the scope of this course, but we will discuss two of its main ingredients: non-abelian gauge fields, or Yang–Mills theories, and spontaneous symmetry breaking through the Higgs mechanism.

In spite of the remarkable successes of the Standard Model, the search for the fundamental laws governing the microscopic world is still very far from being completed. In the Standard Model itself there is still a missing piece, since it predicts a particle, the Higgs boson, which plays a crucial role and which has not yet been observed. LEP, after 11 years of glorious activity, was closed in November 2000, after reaching a maximum center of mass energy of 209 GeV. The new machine, LHC, is now under construction at CERN, and together with the Tevatron collider at Fermilab aims at exploring the TeV ($= 10^3\,\mathrm{GeV} = 10^{12}$ eV) energy range. It is hoped that they will find the Higgs boson and that they will test theoretical ideas like supersymmetry that, if correct, are expected to give observable signals at this energy scale.

Looking much beyond the Standard Model, there is a very substantial reason for believing that we are still far from a true understanding of the fundamental laws of Nature. This is because gravity cannot be included in the conceptual schemes that we have discussed so far. General relativity is incompatible with quantum field theory. From an experimental point of view, at present, this causes no real worry; the energy scale at which quantum gravity effects are expected to become important is so huge (of order 10^{19} GeV) that we can forget them altogether in accelerator experiments.[3] There remains the conceptual need for a new theoretical scheme where these two pillars of modern physics, quantum field theory and general relativity, merge consistently. And, of course, one should also be subtle enough to find situations where this can give testable predictions. A consistent theoretical scheme is perhaps slowly emerging in the form of string theory; but this would lead us very far from the scope of this course.

[3]However, this could change in theories with large extra dimensions. In fact, both in quantum field theory and in string theory, have been devised mechanisms such that some extra dimensions are accessible only to gravitational interactions, and not to electromagnetic, weak or strong interactions. In this case, it turns out that the extra dimensions could even be as large as the millimeter without conflicting with any experimental result, and the huge value 10^{19} GeV of the gravitational scale would emerge from a combination of the large volume of the extra dimensions and a much smaller mass-scale which characterizes the energy where genuine quantum gravity effects set in. This new gravitational mass-scale might even be as low as a few tens of TeV, and in this case it could be within the reach of future particle physics experiments.

1.2 Typical scales in high-energy physics

Before entering into the technical aspects of quantum field theory, it is important to have a physical understanding of the typical scales of atomic and particle physics and to be able to estimate what are the orders of magnitudes involved. Often this can be done just with elementary dimensional considerations, supplemented by some very basic physical inputs. We will therefore devote this section to an overview of order of magnitude estimates in particle physics.

These estimates are much simplified by the use of units $\hbar = c = 1$. To understand the meaning of these units, observe first of all that $\hbar$ and c are universal constants, i.e. they have the same numerical value for all observers. The speed of light has the value $c = 299\,792\,458$ m/s, with no error because, after having defined the unit of time from a particular atomic transition (a hyperfine transition of cesium-133) this value of c is taken as the *definition* of the meter. However, instead of using the meter, we can decide to use a new unit of length (or a new unit of time) defined by the statement that in these units $c = 1$. Then, the velocity v of a particle is measured in units of the speed of light, which is very natural since in particle physics we typically deal with relativistic objects. In these units $0 \leqslant v < 1$ for massive particles, and $v = 1$ for massless particles.

The Planck constant $\hbar$ is another universal constant, and it has dimensions [energy] × [time] or [length] × [momentum] as we see for instance from the uncertainty principle. We can therefore choose units of energy such that $\hbar = 1$. Then all multiplicative factors of $\hbar$ and c disappear from our equations and formally, from the point of view of dimensional analysis,

$$[\text{velocity}] = \text{pure number}\,, \tag{1.2}$$
$$[\text{energy}] = [\text{momentum}] = [\text{mass}]\,, \tag{1.3}$$
$$[\text{length}] = [\text{mass}]^{-1}\,. \tag{1.4}$$

The first two equations follow immediately from $c = 1$ while the third follows from the fact that $\hbar/(mc)$ is a length. Thus all physical quantities have dimensions that can be expressed as powers of mass or, equivalently, as powers of length. For instance an energy density, [energy]/[length]^3, becomes a [mass]^4. Units $\hbar = c = 1$ are called *natural units.*

The fine structure constant $\alpha = e^2/(4\pi\hbar c) \simeq 1/137$ is a pure number, and therefore in natural units the electric charge e becomes a pure number.

To make numerical estimates, it is useful to observe that $\hbar c$, in ordinary units, has dimensions [energy×time]×[velocity] = [energy]×[length]. In particle physics a useful unit of energy is the MeV (= 10^6 eV) and a typical length-scale is the fermi: 1 fm = 10^{-13} cm; one fm is the typical size of a proton. Expressing $\hbar c$ in MeV×fm, one gets

$$\boxed{\hbar c \simeq 200 \text{ MeV fm}\,.} \tag{1.5}$$

(The precise value is 197.326 968 (17) MeV fm.) Then, in natural units, $1\,\text{fm} \simeq 1/(200\,\text{MeV})$. The following examples will show that sometimes we can go quite far in the understanding of physics with just very simple dimensional estimates.

If we want to make dimensional estimates in QED the two parameters that enter are the fine structure constant $\alpha \simeq 1/137$ and the electron mass, $m_e \simeq 0.5$ MeV/c^2. Note that in units $c = 1$ masses are expressed simply in MeV, as energies. We now consider a few examples.

The Compton radius. The simplest length-scale associated to a particle of mass m in its rest frame is its Compton radius, $r_C = 1/m$. In particular, for the electron

$$\boxed{r_C = \frac{1}{m_e} \simeq \frac{200\text{ MeV fm}}{0.5\text{ MeV}} = 4 \times 10^{-11}\,\text{cm}\,.} \tag{1.6}$$

Since r_C does not depend on α, it is the relevant length-scale in situations in which there is no dependence on the strength of the interaction. Historically, r_C made its first appearance in the Compton scattering of X-rays off electrons. Classically, the wavelength of the scattered X-rays should be the same as the incoming waves, since the process is described in terms of forced oscillations. Quantum mechanically, treating the X-rays as photons, we understand that part of the momentum $h\nu$ of the incoming photon is used to produce the recoil of the electron, so the momentum of the outgoing photon is smaller, and its wavelength is larger. The wavelength of the outgoing photon is fixed by energy–momentum conservation, and therefore is independent of α, so the relevant length-scale must be r_C. Indeed, a simple computation gives

$$\lambda' - \lambda = r_C(1 - \cos\theta)\,, \tag{1.7}$$

where λ, λ' are the initial and final X-ray wavelengths and θ is the scattering angle.

The hydrogen atom. Let us first estimate the Bohr radius r_B. The only mass that enters the problem is the reduced mass of the electron–

proton system; since $m_p \simeq 938$ MeV is much bigger than m_e we can identify the reduced mass with m_e, within a precision of 0.05 per cent. Dimensionally, again $r_B \sim 1/m_e$, but now α enters. Clearly, the radius of the bound state is smaller if the interaction responsible for the binding is stronger, while it must go to infinity in the limit $\alpha \to 0$, so α must be in the denominator and it is very natural to guess that $r_B \sim 1/(m_e\alpha)$. This is indeed the case, as can be seen with the following argument: by the uncertainty principle, an electron confined in a radius r has a momentum $p \sim 1/r$. If the electron in the hydrogen atom is non-relativistic (we will verify the consistency of this hypothesis a posteriori) its kinetic energy is $p^2/(2m_e) \sim 1/(2m_e r^2)$. This kinetic energy must be balanced by the Coulomb potential, so at the equilibrium radius $1/(2m_e r^2) \sim \alpha/r$, which indeed gives $r_B \sim 1/(m_e\alpha)$. In principle factors of 2 are beyond the power of dimensional estimates, but here it is quite tempting to observe that the virial theorem of classical mechanics states that, for a potential proportional to $1/r$, at equilibrium the kinetic energy is one half of the absolute value of the potential energy, so we would guess, more precisely, that $1/(2m_e r_B^2) = \alpha/(2r_B)$, i.e.

$$\boxed{r_B = \frac{1}{m_e\alpha} \simeq 0.5 \times 10^{-8}\,\text{cm}\,,} \tag{1.8}$$

which is indeed the definition of the Bohr radius as found in the quantum mechanical treatment. The typical potential energy of the hydrogen atom is then

$$\langle V \rangle \sim V(r_B) = -\frac{\alpha}{r_B} = -m_e\alpha^2\,, \tag{1.9}$$

and, again using the virial theorem, the kinetic energy is

$$E = -\frac{1}{2}V \sim \frac{1}{2}m_e\alpha^2\,. \tag{1.10}$$

This is the kinetic energy of a non-relativistic electron with typical velocity

$$v \sim \alpha\,. \tag{1.11}$$

Since $\alpha \ll 1$, our approximation of a non-relativistic electron is indeed consistent. This of course was expected, since we know that, in a first approximation, the non-relativistic Schrödinger equation gives a good description of the hydrogen atom.

The sum of the kinetic and potential energy is $-(1/2)m_e\alpha^2$ so the binding energy of the hydrogen atom is

$$\text{binding energy} = \frac{1}{2}m_e\alpha^2 \simeq \frac{1}{2}\,0.5\text{MeV}\left(\frac{1}{137}\right)^2 \simeq 13.6\ \text{eV}\,. \tag{1.12}$$

The Rydberg energy is indeed defined as $(1/2)m_e\alpha^2$, and the Schrödinger equation gives the energy levels

$$E_n = -\frac{m_e\alpha^2}{2n^2}\,. \tag{1.13}$$

In QED this is just the first term of an expansion in α; at next order one finds the *fine structure* of the hydrogen atom,

$$E_{n,j} = m_e \left[-\frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4} \left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4} \right) + \dots \right] , \tag{1.14}$$

where j is the total angular momentum and, to be more accurate, the electron mass should be replaced by the reduced mass $m_e m_p/(m_e+m_p)$. We will derive eq. (1.14) in Solved Problem 3.1. The fine structure constant α gets its name from this formula. From eq. (1.11) we understand that, in the hydrogen atom, the expansion in α is the same as an expansion in powers of v, and the fine structure of the hydrogen atom is just the first relativistic correction.

Electron–photon scattering. We want to estimate the cross-section for the scattering of a photon by an electron, which we take initially at rest, $e^-\gamma \to e^-\gamma$. We denote by ω the initial photon energy (in natural units the energy of the photon $E = \hbar\omega$ becomes simply ω). The energy of the final photon is fixed by the initial energy ω and by the scattering angle θ, so the total cross-section (i.e. the cross-section integrated over the scattering angle) can depend only on two energy scales, m_e and ω, and on the dimensionless coupling α. The dependence on α is determined observing that the scattering process takes place via the absorption of the incoming photon and the emission of the outgoing photon. As we will study in detail in Chapters 5 and 7, this is a process of second order in perturbation theory and its amplitude is $O(e^2)$ so the cross-section, which is proportional to the squared amplitude, is $O(e^4)$, i.e. $O(\alpha^2)$. For a generic incoming photon energy ω, we have two different scales in the problem and we cannot go very far with dimensional considerations. Things simplify in the limit $\omega \ll m_e$. In this limit we can neglect ω compared to m_e and we have basically only one mass-scale, m_e. Since the cross-section has dimensions [length]^2, we can estimate $\sigma \sim \alpha^2/m_e^2$. It is therefore useful to define r_0,

$$\boxed{r_0 = \frac{\alpha}{m_e} \simeq 2.8 \times 10^{-13}\,\text{cm}\,,} \tag{1.15}$$

so that the cross-section is $\sigma \sim r_0^2$. The exact computation gives the result

$$\sigma_T = \frac{8}{3}\pi r_0^2 \tag{1.16}$$

and the factor of π is also easily understood, since a cross-section is an effective area, so it is $\sim \pi r_0^2$. The electron–photon cross-section at $\omega \ll m_e$ is known as the Thomson cross-section and can be computed just with classical electrodynamics, since when $\omega \ll m_e$ the photons are well described by a classical electromagnetic field; r_0 is therefore called the *classical electron radius*, and gives a measure of the size of an electron, as seen using classical electromagnetic fields as a probe.

Consider now the opposite limit $\omega \gg m_e$. In this case the cross-section must have a dependence on the energy of the photon and, because of Lorentz invariance, the cross-section integrated over the angles will depend on the energy of the photon through the energy in the center of mass system. If k is the initial four-momentum of the photon and p_e is the initial four-momentum of the electron, the total initial four-momentum is $p = k + p_e$ and the square of the energy in the center of mass is $s = p^2$. In the rest frame of the electron $p_e = (m_e, 0, 0, 0)$ and $k = (\omega, 0, 0, \omega)$, so $s = (m_e+\omega)^2 - \omega^2 = 2m_e\omega + m_e^2$. In the limit $\omega \gg m_e$ we have $s \gg m_e^2$ and we would expect that we can neglect m_e. Then the only energy scale is provided by $\sqrt{s}$, and we would expect that $\sigma \sim \alpha^2/s$. Here however there is a subtlety. In the previous case, $\omega \ll m_e$, we have implicitly assumed that in the limit $\omega \to 0$ the cross-section is finite. This is indeed the case, since in this limit the electromagnetic field can be treated classically, and the classical computation gives a finite answer.[4] If instead $\omega \gg m_e$, we are effectively taking the limit $m_e \to 0$; it turns out that this limit is problematic in QED, and taking $m_e \to 0$ one finds so-called infrared divergences. In fact, from the explicit computation one finds that the correct high-energy limit of the cross-section is

$$\sigma \simeq \frac{2\pi\alpha^2}{s} \log\left(\frac{s}{m_e^2}\right). \tag{1.17}$$

This is an example of the fact that divergences, which are typical of quantum field theory, can spoil naive dimensional analysis. We will examine this issue in a more general context in Section 5.9.

[4] In general, not every quantum computation has a well-defined classical limit; just think of what happens to the black body spectrum when $\hbar \to 0$ (indeed, this example was just the original motivation of Planck for introducing $\hbar$!). However, reinstating $\hbar$ and c explicitly, the classical electron radius is $r_0 = \alpha(\hbar/m_e c) = (e^2/4\pi\hbar c)(\hbar/m_e c)$ and $\hbar$ cancels, so the limit $\hbar \to 0$ is well defined.

In conclusion, we have found three different scales that can be constructed with m_e and α. The largest is $r_B = 1/(m_e\alpha)$ and gives the characteristic size of an electron bound by the Coulomb potential of a proton; $r_C = 1/m_e$ is the characteristic length-scale associated with a free electron in its rest frame, and the smallest, $r_0 = \alpha/m_e$, is associated with classical $e\gamma$ scattering.

Nucleons and strong interactions. Nuclei are bound states of nucleons, i.e. of protons and neutrons, with a radius $r \sim A^{1/3} \times 1$ fm, where A is the total number of nucleons (so that the volume is proportional to A). From the uncertainty principle, a particle confined within 1 fm has a momentum $p \sim 1/(1\,\text{fm}) \simeq 200$ MeV. If the nucleons in the nucleus are non-relativistic, their kinetic energy is

$$E_N \simeq \frac{p_N^2}{2m_N} \simeq 20\,\text{MeV} \tag{1.18}$$

so this must be the typical scale of nuclear binding energies; the typical velocity is

$$v_N \simeq \frac{p_N}{m_N} \simeq 0.2\,. \tag{1.19}$$

This values of v shows that the non-relativistic approximation is roughly correct, but relativistic corrections in nuclei are numerically more important than in atoms. Since the corrections are proportional to v^2 (compare eqs. (1.11) and (1.14)), in nuclei they are of order 4%.

It is also interesting to estimate the analogue of α for the strong interactions. For this we need to know that the nucleon–nucleon strong potential is not Coulomb-like, but rather decays exponentially at large distances,

$$V \simeq -\frac{\alpha_s}{r} e^{-m_\pi r}\,, \tag{1.20}$$

where α_s is the coupling constant of strong interactions and $m_\pi \simeq 140$ MeV is the mass of a particle, the pion, that at length-scales $l \gtrsim 1$ fm can be considered the mediator of the strong interaction (we will derive this result in Section 6.6). Consider for instance a proton–neutron system, which makes a bound state (the nucleus of deuterium) of radius $r \sim 1$ fm. At equilibrium, $(-1/2)V$ must be equal to the kinetic energy $p^2/(2m) \sim 1/(2mr^2)$, where $m \simeq m_p/2$ is the reduced mass of the two-nucleon system (and the $-1/2$ comes again from the virial theorem). Since we already know that the equilibrium radius is at $r \simeq 1$ fm, we find $\alpha_s \sim 2(m_p r)^{-1} \exp\{m_\pi r\}|_{r=1\,\text{fm}} \sim 0.8$. The precise numerical value is not of great significance, since we are making order of magnitude estimates, but anyway this shows that the coupling α_s is not a small number, and strong interactions cannot be treated perturbatively in the same way as QED.[5]

[5] We will see in Section 5.9 that the coupling constants actually are not constant at all, but rather depend on the length-scale at which they are measured. We will see that the correct statement is that the theory of strong interactions, QCD, cannot be treated perturbatively at length-scales $l \gtrsim 1$ fm, while α_s becomes small at $l \ll 1$ fm, and there perturbation theory works well.

Lifetime and cross-sections of strong interactions. Hadrons are defined as particles which have strong interactions. If a particle decays by strong interactions it is possible to estimate its lifetime τ as follows. The quantities that can enter the computation of the lifetime are the coupling α_s, the masses of the particles involved, and the typical interaction radius of the strong interactions. However, these particles have typical masses in the GeV range, and the interaction range of the strong interaction $\sim 1\text{fm} \simeq (200\text{MeV})^{-1}$. Then all energy scales in the problem are between a few hundred MeV and a few GeV, so in a first approximation we can say that the only length-scale in the problem is of the order of the fermi. Furthermore, we have seen that $\alpha_s = O(1)$. This means that, in order of magnitude, the lifetimes of particles which decay by strong interactions are in the ballpark of $\tau \sim 1\,\text{fm}/c \sim 3 \times 10^{-24}$ s. Particles with such a small lifetime only show up as peaks in a plot of a scattering cross-section against the energy, and are called resonances, since the mechanism that produces the peak is conceptually the same as the resonance in classical mechanics (we will discuss resonances in detail in Section 6.5). The width Γ of the peak is related to the lifetime by $\Gamma = \hbar/\tau$ or, in natural units,

$$\Gamma = \frac{1}{\tau} \sim \frac{1}{1\,\text{fm}} \simeq 200\,\text{MeV}\,. \tag{1.21}$$

We can estimate similarly the typical cross-sections of processes mediated by strong interactions. Since a cross-section is an effective area, we must typically have $\sigma \sim \pi\,(1\,\text{fm})^2 \sim 3 \times 10^{-26}\text{cm}^2$. A common unit for cross-sections is the barn, 1 barn $= 10^{-24}\,\text{cm}^2$. Therefore a typical strong interactions cross-section, in the absence of dynamical phenomena like resonances, is of the order of 30 millibarns. Here we have implicitly assumed that the particles are relativistic, i.e. their relative speed is close to one. Otherwise we must take into account that the relevant length-scale for a particle of mass m and velocity $v \ll 1$ is given by the De Broglie wavelength $\lambda = 1/(mv) \gg 1/m$, and a typical nuclear cross-section for slow particles, in the absence of resonances, is of the order $\sigma \sim \pi\lambda^2$, see Exercise 1.3.

Electroweak decays. Leptons do not have strong interactions and either are stable or decay through electroweak interactions. Furthermore, strong interactions obey a number of conservation laws, which result in the fact that also many hadrons cannot decay via the strong interaction; in this case they decay through electroweak interactions (except for the proton, which in the Standard Model is stable) and their lifetime is considerably longer than the typical lifetimes $\tau \sim 10^{-24}$ s of strong decays. Weak decays span a broad range of lifetimes because they depend on quite different mass-scales: the electroweak scale, the mass of the decaying particle, and the masses of the decay products. While in the case of hadronic resonances the scales which are involved are all between a few hundred MeV and a few GeV, for weak decays these scales can be very different from each other: the electroweak scale is $O(100)$ GeV, while the masses of the decaying particle or of the decay products can be anywhere between zero (for the photon) or less than a few eV (for the electron neutrino) up to hundreds of GeV. Furthermore the electroweak coupling constants are not of order one. Rather, the electromagnetic coupling is $\alpha \sim 1/137 \simeq 0.007$ while, as we will discuss in Chapter 8, weak interactions are characterized by two coupling constants $g^2/(4\pi)$ and $\bar{g}^2/(4\pi)$ both numerically of order 0.1. For these reasons the electroweak lifetimes, even in order of magnitude, vary from case to case. Some examples are given in Table 1.1.

Table 1.1 Examples of electroweak decays. In the right column we give the lifetime of the decaying particle and in the left column its main decay mode. Observe the broad range of lifetimes. For lifetimes so small as for the Z^0, it is more convenient to give the decay width. For the Z^0, the full width is $\Gamma = 2.4952(23)$ GeV.

main mode			lifetime (sec)
n	$\rightarrow$	$pe^-\bar{\nu}_e$	$0.8857(8) \times 10^3$
μ^-	$\rightarrow$	$e^-\bar{\nu}_e\nu_\mu$	$2.19703(4) \times 10^{-6}$
π^+	$\rightarrow$	$\mu^+\nu_\mu$	$2.6033(5) \times 10^{-8}$
Λ^0	$\rightarrow$	$p\pi^-$	$2.632(20) \times 10^{-10}$
K_S^0	$\rightarrow$	$\pi^+\pi^-$	$0.8958(6) \times 10^{-10}$
π^0	$\rightarrow$	$\gamma\gamma$	$0.84(6) \times 10^{-16}$
Σ^0	$\rightarrow$	$\Lambda\gamma$	$0.74(7) \times 10^{-19}$
Z^0	$\rightarrow$	hadrons	$2.6379(24) \times 10^{-25}$

The lifetime can be written as

$$\tau = \frac{\hbar}{\Gamma} = \frac{\hbar}{\sum_i \Gamma_i} \tag{1.22}$$

where in the last equality the sum runs over all decay channels. Γ is called the full width, while the Γ_i are the partial widths relative to the decay mode labeled by i. In the first column of Table 1.1 we give the dominant decay mode, i.e. the mode with the largest partial width. In the second column we give the lifetime, i.e. the inverse of the full width. The quantity Γ_i/Γ is called the branching ratio of the mode labeled by i. We will compute explicitly many weak decays in the Solved Problems section of Chapter 8.

The Planck mass. Using simple dimensional estimates we can also understand the statement made at the end of Section 1.1 that, in the realm of particle physics, gravity enters into play only at huge energies. Comparing the Newton potential $V = -G_N m^2/r$ with a Coulomb potential $V = -e^2/4\pi r = -(\alpha\hbar c)/r$, we see that G_N times a mass squared has the dimensions of $\hbar c$. Therefore from the fundamental constants $\hbar, c, G_N$ we can build a mass-scale

$$M_{\mathrm{Pl}} = \sqrt{\frac{\hbar c}{G_N}}\,, \tag{1.23}$$

known as the Planck mass, whose numerical value is $M_{\mathrm{Pl}} \simeq 1.2 \times 10^{19}\,\mathrm{GeV}/c^2$. In natural units, then, $G_N = 1/M_{\mathrm{Pl}}^2$ and we see, comparing the Newton and Coulomb laws, that the gravitational analogue of the fine structure constant is $(m/M_{\mathrm{Pl}})^2$. More precisely, since in general relativity any form of energy is a source for the gravitational field, particles with an energy E have an effective gravitational coupling

$$\alpha_G = \frac{E^2}{M_{\mathrm{Pl}}^2}\,. \tag{1.24}$$

At the typical energies of particle physics, say $E \sim 1$ GeV, we have $\alpha_G \sim 10^{-38}$ and gravity is completely irrelevant. In the realm of particle physics, gravity becomes important only at energies comparable to the Planck scale. These considerations only apply to the microscopic domain. On the macroscopic scale, gravity can become more important than electric interactions because it is always attractive, so it has a cumulative effect, while on a large scale the electrostatic forces are screened by the formation of electrically neutral objects, and the residual force decreases faster than $1/r^2$.

Since M_{Pl} provides a natural mass-scale, in quantum gravity it is customary to use units in which not only $\hbar$ and c but also M_{Pl} are set equal to one. These are called Planck units, and in these units all physical quantities are dimensionless. We will not use them in this book.

Further reading

- A historical introduction to quantum field theory is given in Weinberg (1995), Chapter 1.
- The standard compilation of experimental data for high-energy physics is the Review of Particle Physics of the Particle Data Group. Unless explicitly stated otherwise, our experimental data are taken from the 2004 edition, S. Eidelman *et al.*, *Phys. Lett.* B592, 1 (2004), also available on-line at http://pdg.lbl.gov.
- Precision measurements are a fascinating field by themselves; the experimentally minded student might enjoy browsing the detailed article by F. J. M. Farley and E. Picasso, *The muon g-2 experiment*, in T. Kinoshita ed., *Quantum Electrodynamics*, World Scientific, Singapore 1990. Recently the measure of the $g-2$ of the muon has been further improved by an experiment in Brookhaven, see the link http://www.g-2.bnl.gov/
- A well-written popular book, which gives a flavor of modern research in quantum gravity and string theory is B. Greene, *The elegant universe: superstrings, hidden dimensions, and the quest for the ultimate theory*, Norton, New York 1999.
- QFT is a domain where there can be an interplay between frontier research in theoretical physics and in pure mathematics, and in the last decades this has generated important advances in both fields. The physicist who wishes an introduction to the ap-

plication to physics of important concepts of geometry and topology (like cohomology groups, complex manifolds, fibre bundles, characteristic classes, etc.) can consult, for instance, Nakahara (1990). These concepts find many applications in the theory of non-abelian gauge fields and in string theory. Conversely, the mathematician interested in the mathematical applications of QFT, supersymmetry and string theory is referred to P. Deligne *et al.* eds., *Quantum Fields and Strings: A Course for Mathematicians*, AMS IAS 1999.

Exercises

(1.1) The Universe is permeated by a thermal background of electromagnetic radiation at a temperature $T = 2.725(1)$ K (the cosmic microwave background radiation, or CMB). Estimate with dimensional arguments the energy density of this gas of photons and compare it with the critical density for closing the Universe, $\rho_c \sim 0.5 \times 10^{-5}\text{GeV/cm}^3$.

[Hint: a useful mnemonic for k_B is given by the fact that, at room temperature $T = 300$ K, $k_BT \simeq (1/40)$ eV. In the energy density, the numerical constant in front of $(k_BT)^4$ turns out to be $(\pi^2/30)g(T)$, where $g(T)$ is of the order of the number of particles which are relativistic at a temperature T, i.e. which have $m \ll T$. With $T \simeq 2.7$ K, only the photon and at most three neutrinos are relativistic and $g(T)$ is between 3 and 4. Then, for the purpose of this exercise, the only thing that matters is that the constant $(\pi^2/30)g(T)$ is of order one.]

(1.2) Model the Sun as an ionized plasma of electrons and protons, with an average temperature $T \simeq 4.5 \times 10^6$ K and an average mass density $\rho \simeq 1.4\,\text{gm/cm}^3$. Estimate the mean free path of photons in the Sun's interior, and compare the contribution to the mean free path coming from the scattering on electrons with that from the scattering on protons. Knowing that the radius of the Sun is $R_\odot \simeq 6.96\times10^{10}$ cm, estimate the total time that a photon takes to escape from the Sun.

[Hint: recall that the mean free path l of a particle scattering off an ensemble of targets with number density (i.e. particles per unit volume) n and cross-section σ is

$$l = \frac{1}{n\sigma} \tag{1.25}$$

or, if there are different species of targets, $l = 1/\sum_i n_i\sigma_i$.]

(1.3) Estimate the cross-section for a non-relativistic neutron with kinetic energy $E \sim 1$ MeV, scattering on a proton at rest.