1
Operator spaces

In this chapter, we present quickly the background results about operator spaces which we shall need, and also establish some notation which will be used throughout this book. The reader with a little mathematical maturity could use this chapter as a minicourse on the basics of operator space theory. Fortunately the lengthy proofs here usually belong to the very well-known results (such as Ruan's theorem, or the extension/characterization theorems for completely positive or completely bounded maps). Thus with the exception of a few such well-known proofs (which may be found in [149, 314, 337, 385, 102]), we can be quite self-contained. Warning: our proofs in this chapter are often only a good sketch, and some things are left as exercises. The reader should also feel free to skim through this chapter, returning later for a specific definition or fact (we try to be conscientious in later chapters about referencing these by number). Those mainly interested in the general theory of operator spaces, should consult the fine aforementioned texts for a more comprehensive and leisurely development. And of course usually the original papers contain much additional material.

We will take for granted facts found in any basic graduate level functional analysis text. For example, we assume that the reader is comfortable with basic spectral theory, the very basics of the theory of C^*-algebras and Banach algebras, and standard facts about the various topologies in Banach spaces or dual spaces. Much of this may be found in the Appendix, together with a few of the unexplained terms below.

1.1 NOTATION AND CONVENTIONS

1.1.1 Our set notation and function notation is standard. We use A^c for the complement of a set A. The term 'scalar' denotes a number in the complex field $\mathbb{C}$. We use n, m, i, j, k for integers, and I, J or α, β, γ for cardinal numbers. Vector spaces are almost always over the field $\mathbb{C}$ unless stated to the contrary. The usual basis of $\mathbb{C}^n$ or ℓ^2 is written as $(e_i)_i$, and we use this notation too in the other ℓ^p sequence spaces. We write I_E, or sometimes I when there is no confusion, for the 'identity map' on a vector space E. An isomorphism, at the very least, is always assumed to be linear, one-to-one, and surjective. If $T\colon E \to F$, and if $W \subset E$ is a linear subspace, then we write $T_{|W}$ for the map from W to F obtained by restricting T to W. We often use the symbol q or q_W for the canonical surjection from E onto E/W. We write $x\dot{+}W$, or sometimes $\dot{x}$, for the class of x in E/W, thus $x\dot{+}W = q_W(x)$.

If E is a normed space, we write $\mathrm{Ball}(E)$ for the set $\{x \in E : \|x\| \leq 1\}$. Expressions such as 'norm closed', 'norm closure', or '$x_n \to x$ in norm' (or simply 'closed', 'closure', or '$x_n \to x$'), mean of course 'with respect to the norm topology'. All topological spaces are assumed to be Hausdorff. We use standard notation for the standard examples, for example, $C(\Omega)$ is the Banach space of scalar valued continuous functions on a compact space Ω. In the literature these are often called '$C(K)$-spaces', and of course are exactly the commutative unital C^*-algebras (see A.5.4). We use the letters H, K, L for Hilbert spaces. Thus if these letters appear in the text without explanation, they will always be Hilbert spaces. We write $B(E, F)$ for the space of bounded linear operators from E to F, and $B(E) = B(E, E)$. Indeed whenever $\mathcal{C}(X, Y)$ is a class of operators then we use $\mathcal{C}(X)$ for $\mathcal{C}(X, X)$. We write E^* for the dual space of E, namely $E^* = B(E, \mathbb{C})$, and we often write E_* for a predual of E (if such exists). We write $i_E \colon E \to E^{**}$ for the canonical embedding, but will often simply think of E as a subspace of E^{**}. We abbreviate 'weak*' to 'w^*' usually. Thus we write w^*-continuous, w^*-topology, w^*-closure, etc. Thus $\overline{\mathcal{S}}^{w^*}$ denotes the w^*-closure of a set $\mathcal{S}$. We say that a net of maps $T_t \colon E \to F$ converges *strongly* (or *point-norm*) if $T_t(x) \to T(x)$ in the norm topology of F for all $x \in E$. If F is a dual space then $T_t \to T$ *point-w^** if $T_t(x) \to T(x)$ in the w^*-topology of F for all $x \in E$. A multilinear map between dual spaces is said to be *separately w^*-continuous* if whenever one fixes all but one of the variables, then the map is w^*-continuous in the remaining variable. We recommend that the reader review the facts about the w^*-topology presented in the first sections of the Appendix.

An operator T between normed spaces, with $\|T\| \leq 1$, is called a *contraction*. A *quotient map* $T \colon E \to F$ is a linear map which maps the 'open unit ball of E' onto the 'open unit ball of F'. A *projection* or *idempotent* on a space E is a map $P \colon E \to E$ satisfying $P \circ P = P$. However if E is a Hilbert space then we will mean more, indeed for an operator on a Hilbert space, or more generally for an element of an operator algebra, we always use the term *projection* to mean an orthogonal (i.e. selfadjoint) idempotent. If K is a closed linear subspace of a Hilbert space H then P_K is the canonical projection from H onto K.

1.1.2 For emphasis, we list separately here some of our major conventions. First, we usually suppose that all of our normed spaces are complete. This is not a serious restriction, since the completion of an operator space is again an operator space; and the 'incomplete' versions of most results 'pass to the completion' without difficulty. We make the 'completeness' assumption mostly to avoid having to be constantly making annoying and repetitious remarks about results 'passing to the completion'. Another convention is our use of the notation XY for sets X, Y. Assume that we have a pairing $X \times Y \to E$ where E is a Banach space. Write this pairing as the map $(x, y) \mapsto xy$. Then XY usually denotes the *closure* in the norm topology in E of the linear span of the xy, for $x \in X$ and $y \in Y$. We write $\mathrm{Span}(XY)$ if we are not taking the closure here.

See also A.6.4 for some important related facts. There is an exception to this notation; if K is a subset of a Hilbert space H and if $D \subset B(H,L)$ is a set of operators then we use $[DK]$ for the norm closure in L of the span of terms $x\zeta$ for $x \in D, \zeta \in K$.

If X is a subspace of $B(K,H)$ or of a C^*-algebra, then we often use the symbol $X^\star$ (also written as X^* when there is no possible confusion with the dual space) for the set of 'adjoints' or 'involutions' $\{x^* : x \in X\}$.

1.1.3 (Matrix notation) Fix $m, n \in \mathbb{N}$. If X is a vector space, then so is $M_{m,n}(X)$, the set of $m \times n$ matrices with entries in X. This may also be thought of as the algebraic tensor product $M_{m,n} \otimes X$, where $M_{m,n} = M_{m,n}(\mathbb{C})$. We write I_n for the identity matrix of $M_n = M_{n,n}$. We write $M_n(X) = M_{n,n}(X)$, $C_n(X) = M_{n,1}(X)$ and $R_n(X) = M_{1,n}(X)$.

If x is a matrix, then x_{ij} or $x_{i,j}$ denotes the i-j entry of x, and we write x as $[x_{ij}]$ or $[x_{i,j}]_{i,j}$. We write $(E_{ij})_{ij}$ for the usual (matrix unit) basis of $M_{m,n}$ (we allow m, n infinite here too). We write $A \mapsto A^t$ for the transpose on $M_{m,n}$, or more generally on $M_{m,n}(X)$. We will frequently meet large matrices with row and column indexing that is sometimes cumbersome. For example, a matrix $[a_{(i,k,p),(j,l,q)}]$ is indexed on rows by (i,k,p) and on columns by (j,l,q), and may also be written as $[a_{(i,k,p),(j,l,q)}]_{(i,k,p),(j,l,q)}$ if additional clarity is needed.

1.1.4 The Hilbert space direct sum will be written as $\oplus^2$, or simply $\oplus$ (but we use the latter for some other kinds of direct sums too). We also write $H^{(\alpha)}$ or $\ell^2_\alpha(H)$ for the Hilbert space direct sum of α copies of H. Here α is a cardinal. This is called a *multiple* of H. The Hilbert space tensor product is denoted $H \otimes^2 K$. If S, T are operators on H and K respectively, then we write $S \otimes T$ for the usual operator on $H \otimes^2 K$ taking a rank one tensor $\zeta \otimes \eta$ in $H \otimes K$ to $S(\zeta) \otimes T(\eta)$. In particular, $S \otimes I_K$ is often called a *multiple* of S. Indeed, if K is identified with ℓ^2_α for some cardinal α, then we may unitarily identify $H \otimes^2 K$ with $\ell^2_\alpha(H)$, and $S \otimes I_K$ with S^α. Here $S^\alpha((\zeta_i)) = (S\zeta_i)$, for $(\zeta_i) \in \ell^2_\alpha(H)$. The commutant of a subset $\mathcal{S} \subset B(H)$ is written as $\mathcal{S}'$ or $[\mathcal{S}]'$. The *C^*-identity* is the statement

$$\|T^*T\| = \|T\|^2,$$

valid for any bounded operator T between Hilbert spaces, or any element of a C^*-algebra. We write $S^p(K,H)$ for the Schatten p class (see also A.1.2 and A.1.3). If $H = K$ is n-dimensional then we write this as S^p_n, thus S^1_n is the dual space of M_n. We use WOT for the weak operator topology (see A.1.4), although we usually prefer to use the (finer) w^*-topology ($= \sigma$-weak topology, see A.1.2). Which of these two topologies one uses is often a matter of taste, in the situations we consider. Very frequently, we will need the *polarization identity*. We state one form of it: Suppose that E and F are vector spaces, and that $\Psi \colon E \times E \to F$ is linear in the second variable and conjugate linear in the first variable. Then

$$\Psi(y,x) \;=\; \frac{1}{4}\sum_{k=0}^{3} i^k\, \Psi(x + i^k y, x + i^k y), \qquad x, y \in E. \tag{1.1}$$

This is frequently applied when $E = F$ is a $*$-algebra, and $\Psi(x, y) = x^* y$.

1.1.5 We also use some basic notions from algebra, such as the definitions of modules, algebras, ideals, direct sum, tensor product, etc. These may be found in any graduate algebra text. Our spaces, of course, usually have extra functional analytic structure, and in particular possess a (complete) norm. If A is an algebra, then $M_n(A)$ is also an algebra, if one uses the usual formula for multiplying matrices. We usually refer to a closed two-sided ideal of a normed algebra simply as an 'ideal'. One unusual usage: we use the term *unital-subalgebra* for a subalgebra of a unital algebra A containing the unit (identity) of A. Similarly, a *unital-subspace* is a subspace containing the 'unit' of the superspace. A *unital map* is one that takes the unit to the unit.

We use the very basics of the language of categories, such as the notion of object, morphism, and functor. The main categories we are interested in here are those of Banach spaces and bounded linear maps, operator spaces and completely bounded linear maps, operator algebras and completely contractive homomorphisms, C^*-algebras and $*$-homomorphisms, and operator modules and completely bounded module maps. These notions will be introduced in detail later. However it is worth saying that each of these categories (and any others we shall meet) has its own notion of 'isomorphism' (i.e. when we consider two objects as being essentially the same), subobject, embedding, quotient, quotient map, direct sum, etc. When we use one of these words in later chapters, it is usually understood to be with reference to the category that we are working in. For example, in Chapter 2 we may simply write '$A \cong B$', or '$A \cong B$ as operator algebras', and say that 'A is isomorphic to B', when we really mean that there is a surjective algebra homomorphism between them which is completely isometric (defined below). Or we may write $A \hookrightarrow B$ to indicate that A is 'embedded' in B in the suitable sense of that chapter. For example, in Chapter 2 it means that there is a completely isometric algebra homomorphism from A to B.

1.2 BASIC FACTS, CONSTRUCTIONS, AND EXAMPLES

1.2.1 (Completely bounded maps) Suppose that X and Y are vector spaces and that $u \colon X \to Y$ is a linear map. For a positive integer n, we write u_n for the associated map $[x_{ij}] \mapsto [u(x_{ij})]$ from $M_n(X)$ to $M_n(Y)$. This is often called the (nth) *amplification* of u, and may also be thought of as the map $I_{M_n} \otimes u$ on $M_n \otimes X$. Similarly one may define $u_{m,n} \colon M_{m,n}(X) \to M_{m,n}(Y)$. If each matrix space $M_n(X)$ and $M_n(Y)$ has a given norm $\|\cdot\|_n$, and if u_n is an isometry for all $n \in \mathbb{N}$, then we say that u is *completely isometric*, or is a *complete isometry*. Similarly, u is *completely contractive* (resp. *is a complete quotient map*) if each u_n is a contraction (resp. takes the open ball of $M_n(X)$ onto the open ball of $M_n(Y)$). A map u is *completely bounded* if

$$\|u\|_{\mathrm{cb}} \stackrel{def}{=} \sup\big\{\|[u(x_{ij})]\|_n : \|[x_{ij}]\|_n \leq 1, \text{ all } n \in \mathbb{N}\big\} < \infty.$$

Compositions of completely bounded maps are completely bounded, and one has the expected relation $\|u \circ v\|_{\text{cb}} \leq \|u\|_{\text{cb}}\|v\|_{\text{cb}}$. If $u\colon X \to Y$ is a completely bounded linear bijection, and if its inverse is completely bounded too, then we say that u is a *complete isomorphism*. In this case, we say that X and Y are *completely isomorphic* and we write $X \approx Y$.

1.2.2 (Operator spaces) If $m, n \in \mathbb{N}$, and K, H are Hilbert spaces, then we always assign $M_{m,n}(B(K,H))$ the norm (written $\|\cdot\|_{m,n}$) ensuring that

$$M_{m,n}(B(K,H)) \cong B(K^{(n)}, H^{(m)}) \quad \text{isometrically} \tag{1.2}$$

via the natural algebraic isomorphism. Recall from 1.1.4 that $H^{(m)} = \ell^2_m(H)$ is the Hilbert space direct sum of m copies of H, for example.

A *concrete operator space* is a (usually closed) linear subspace X of $B(K,H)$, for Hilbert spaces H, K (indeed the case $H = K$ usually suffices, via the canonical inclusion $B(K,H) \subset B(H \oplus K)$). However we will want to keep track too of the norm $\|\cdot\|_{m,n}$ that $M_{m,n}(X)$ inherits from $M_{m,n}(B(K,H))$, for all $m, n \in \mathbb{N}$. We write $\|\cdot\|_n$ for $\|\cdot\|_{n,n}$; indeed when there is no danger of confusion, we simply write $\|[x_{ij}]\|$ for $\|[x_{ij}]\|_n$. An *abstract operator space* is a pair $(X, \{\|\cdot\|_n\}_{n\geq 1})$, consisting of a vector space X, and a norm on $M_n(X)$ for all $n \in \mathbb{N}$, such that there exists a linear complete isometry $u\colon X \to B(K,H)$. In this case we call the sequence $\{\|\cdot\|_n\}_n$ an *operator space structure* on the vector space X. An *operator space structure* on a normed space $(X, \|\cdot\|)$ will usually mean a sequence of matrix norms as above, but with $\|\cdot\| = \|\cdot\|_1$.

Clearly subspaces of operator spaces are again operator spaces. We often identify two operator spaces X and Y if they are completely isometrically isomorphic. In this case we often write '$X \cong Y$ completely isometrically', or say '$X \cong Y$ *as operator spaces*'. Sometimes we simply write $X = Y$.

1.2.3 (C^*-algebras) If A is a C^*-algebra then the $*$-algebra $M_n(A)$ has a unique norm with respect to which it is a C^*-algebra, by A.5.8. With respect to these matrix norms, A is an operator space. This may be seen by noting that $M_n(A)$ corresponds to a closed $*$-subalgebra of $B(H^{(n)})$, when A is a closed $*$-subalgebra of $B(H)$. We call this the *canonical operator space structure* on a C^*-algebra. If the C^*-algebra A is commutative, then $A = C_0(\Omega)$ for a locally compact space Ω, and then these matrix norms are determined via the canonical isomorphism $M_n(C_0(\Omega)) = C_0(\Omega; M_n)$. Explicitly, if $[f_{ij}] \in M_n(C_0(\Omega))$, then:

$$\|[f_{ij}]\|_n \;=\; \sup_{t\in\Omega} \big\|[f_{ij}(t)]\big\|. \tag{1.3}$$

To see this, note that by the above one only needs to verify that (1.3) does indeed define a C^*-norm on $M_n(C_0(\Omega))$.

Proposition 1.2.4 *For a homomorphism $\pi\colon A \to B$ between C^*-algebras, the following are equivalent:* (i) *π is contractive,* (ii) *π is completely contractive, and* (iii) *π is a $*$-homomorphism. If these hold, then $\pi(A)$ is closed, and π is a* *complete quotient map onto $\pi(A)$; moreover π is one-to-one if and only if it is completely isometric.*

Proof Apply A.5.8 to the 'amplifications' π_n. $\square$

1.2.5 (Norm of a row or column) Suppose that A is a C^*-algebra, or a space of the form $B(K,H)$, for Hilbert spaces H,K. If X is a subspace of A, and if $x_1,\dots,x_n \in X$, then we have

$$\left\|[x_1 \ \cdots \ x_n]\right\|_{R_n(X)} = \Big\|\sum_{k=1}^n x_k x_k^*\Big\|^{\frac{1}{2}} \quad \text{and} \quad \left\|\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right\|_{C_n(X)} = \Big\|\sum_{k=1}^n x_k^* x_k\Big\|^{\frac{1}{2}}, \tag{1.4}$$

where the product and involution mean the obvious thing (in the ambient superspace). Indeed this follows from the C^*-identity (see 1.1.4).

1.2.6 (Maps into a commutative C^*-algebra) If $[a_{ij}] \in M_n$ then

$$\|[a_{ij}]\| \;=\; \sup\Big\{\Big|\sum_{ij} a_{ij} z_j \overline{w_i}\Big| \;:\; z=[z_j],\ w=[w_i] \in \mathrm{Ball}(\ell_n^2)\Big\}.$$

It is easy to see that this also holds, but with '$=$' replaced by '$\geq$', and $|\cdot|$ replaced by $\|\cdot\|$, if $a_{ij} \in B(H)$. Using these formulae, it is a simple exercise to see that any continuous functional $\varphi\colon X \to \mathbb{C}$ on an operator space X is completely bounded, with $\|\varphi\| = \|\varphi\|_{\mathrm{cb}}$. From this and equation (1.3), it follows that $\|u\| = \|u\|_{\mathrm{cb}}$ for any bounded linear map u from an operator space into a commutative C^*-algebra.

1.2.7 The following trivial principle is used very often: If we are given complete contractions $v\colon X \to Y$ and $u\colon Y \to Z$, and if uv is a complete isometry (resp. complete quotient map) then v is a complete isometry (resp. u is a complete quotient map). If, further, $Z = X$ and uv and vu are both equal to the identity map, then both u and v are surjective complete isometries, and $u = v^{-1}$.

Theorem 1.2.8 (Haagerup, Paulsen, Wittstock) *Suppose that X is a subspace of a C^*-algebra B, that H and K are Hilbert spaces, and that $u\colon X \to B(K,H)$ is a completely bounded map. Then there exists a Hilbert space L, a $*$-representation $\pi\colon B \to B(L)$ (which may be taken to be unital if B is unital), and bounded operators $S\colon L \to H$ and $T\colon K \to L$, such that $u(x) = S\pi(x)T$ for all $x \in X$. Moreover this can be done with $\|S\|\|T\| = \|u\|_{\mathrm{cb}}$.*

In particular, if $\varphi \in \mathrm{Ball}(X^*)$, and if B is as above, then there exist L,π as above, and unit vectors $\zeta,\eta \in L$, with $\varphi = \langle \pi(\cdot)\zeta,\eta\rangle$ on X.

The very last line clearly follows from the lines above it, and 1.2.6. Also note that conversely, any linear map u of the form $u = S\pi(\cdot)T$ as above, is completely bounded with $\|u\|_{\mathrm{cb}} \leq \|S\|\|T\|$. This is an easy exercise using Proposition 1.2.4.

We omit the well-known proof of Theorem 1.2.8 (see the cited texts above).

1.2.9 (Injective spaces) An operator space Z is said to be *injective* if for any completely bounded linear map $u\colon X \to Z$ and for any operator space Y containing X as a closed subspace, there exists a completely bounded extension $\hat{u}\colon Y \to Z$ such that $\hat{u}_{|X} = u$ and $\|\hat{u}\|_{\mathrm{cb}} = \|u\|_{\mathrm{cb}}$. A similar definition exists for Banach spaces. Thus an operator space (resp. Banach space) is injective if and only if it is an 'injective object' in the category of operator (resp. Banach) spaces and completely contractive (resp. contractive) linear maps.

The following is 'contained' in Theorem 1.2.8 (and the remark after it).

Theorem 1.2.10 *If H and K are Hilbert spaces then $B(K,H)$ is an injective operator space.*

Recall that one version of the Hahn–Banach theorem may be formulated as the statement that $\mathbb{C}$ is injective (as a Banach space). Thus 1.2.10 is a 'generalized Hahn–Banach theorem'.

Corollary 1.2.11 *An operator space is injective if and only if it is linearly completely isometric to the range of a completely contractive idempotent map on $B(H)$, for some Hilbert space H.*

Proof ($\Rightarrow$) Supposing $X \subset B(H)$, extend I_X to a map from $B(H)$ to X.
($\Leftarrow$) Follows from 1.2.10 and an obvious diagram chase. $\square$

1.2.12 (Properties of matrix norms) If K, H are Hilbert spaces, and if X is a subspace of $B(K,H)$, then there are certain well-known properties satisfied by the matrix norms $\|\cdot\|_{m,n}$ described in 1.2.2. For example, adding (or dropping) a row of zeros or column of zeros does not change the norm of a matrix of operators. By this principle we really only need to specify the norms for square matrices, that is, the case $m = n$ above. Also, switching two rows (or two columns) of a matrix of operators does not change its norm. From this we derive another useful property. Namely, the canonical algebraic isomorphisms

$$M_n(M_m(X)) \cong M_m(M_n(X)) \cong M_{mn}(X) \tag{1.5}$$

are isometric, and hence, by iteration, completely isometric. Thus if X is an operator space then so is $M_n(X)$ (or $M_{m,n}(X)$).

As an exercise in operator theory, one may verify that for such X we have:

(R1) $\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|$, for all $n \in \mathbb{N}$ and all $\alpha, \beta \in M_n$, and $x \in M_n(X)$ (where multiplication of an element of $M_n(X)$ by an element of M_n is defined in the obvious way).

(R2) For all $x \in M_m(X)$ and $y \in M_n(X)$, we have

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}.$$

Conditions (R1) and (R2) above are often called *Ruan's axioms*. *Ruan's theorem* asserts that (R1) and (R2) characterize operator space structures on a vector space. This result is fundamental to our subject in many ways. At the most pedestrian level, it is used frequently to check that certain abstract constructions with operator spaces remain operator spaces. At a more sophisticated level, it is the foundational and unifying principle of operator space theory.

Theorem 1.2.13 (Ruan) *Suppose that X is a vector space, and that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(X)$. Then X is linearly completely isometrically isomorphic to a linear subspace of $B(H)$, for some Hilbert space H, if and only if conditions* (R1) *and* (R2) *above hold.*

1.2.14 (Quotient operator spaces) If $Y \subset X$ is a closed linear subspace of an operator space, then using Ruan's theorem one can easily check that X/Y is an operator space with matrix norms coming from the identification $M_n(X/Y) \cong M_n(X)/M_n(Y)$. Explicitly, these matrix norms are given by the formula $\|[x_{ij}\dot{+}Y]\|_n = \inf\{\|[x_{ij} + y_{ij}]\|_n : y_{ij} \in Y\}$. Here $x_{ij} \in X$.

1.2.15 (Factor theorem) If $u \colon X \to Z$ is completely bounded, and if Y is a closed subspace of X contained in $\mathrm{Ker}(u)$, then the canonical map $\tilde{u} \colon X/Y \to Z$ induced by u is also completely bounded, with $\|\tilde{u}\|_{\mathrm{cb}} = \|u\|_{\mathrm{cb}}$. If $Y = \mathrm{Ker}(u)$, then u is a complete quotient map if and only if $\tilde{u}$ is a completely isometric isomorphism. Indeed this follows exactly the usual Banach space case.

1.2.16 (Operator seminorms) An *operator seminorm structure* on a vector space X is a sequence $\rho = \{\rho_n\}_{n=1}^{\infty}$, where ρ_n is a seminorm on $M_n(X)$, satisfying axioms (R1) and (R2) discussed in 1.2.12. In this case, and if N is defined to be $\{x \in X : \rho_1(x) = 0\}$, by (R1) we have that the kernel of ρ_n is $M_n(N)$, and ρ induces matrix norms on X/N in the obvious fashion. By Ruan's theorem, (the completion of) X/N is then an operator space.

Let X be a vector space, and let $\mathcal{F} = \{T_i : i \in I\}$ be a set of linear maps, where T_i maps X into an operator space Z_i, for each $i \in I$. We suppose that $\sup_i \|T_i(x)\| < \infty$, for all $x \in X$. Let $N = \cap_i \mathrm{Ker}(T_i)$. For each $n \in \mathbb{N}$, we define a seminorm on $M_n(X)$ by

$$[x_{pq}] \longmapsto \sup_i \|[T_i(x_{pq})]\|.$$

This is fairly clearly an operator seminorm structure for X. Using the facts in the last paragraph, these seminorms become matrix norms on X/N, and with these norms (the completion of) X/N is an operator space. Similarly, if I is a directed set then the expressions $\limsup_i \|[T_i(x_{pq})]\|$ define an operator seminorm structure on X. This yields an operator space as before.

1.2.17 (The ∞-direct sum) This is the simplest direct sum of a family of operator spaces $\{X_\lambda : \lambda \in I\}$, and we will write this operator space as $\oplus_\lambda X_\lambda$ (or $\oplus_\lambda^\infty X_\lambda$ if more clarity is needed). If $I = \{1, \ldots, n\}$ then we usually write this sum as $X_1 \oplus^\infty \cdots \oplus^\infty X_n$. If $X_\lambda \subset B(H_\lambda)$ then $\oplus_\lambda X_\lambda$ may be regarded as the obvious subspace of $B(\oplus_\lambda^2 H_\lambda)$. A tuple (x_λ) is in $\oplus_\lambda^\infty X_\lambda$ if and only if $x_\lambda \in X_\lambda$

for all λ, and $\sup_\lambda \|x_\lambda\| < \infty$. We may identify $M_n(\oplus_\lambda X_\lambda)$ with $\oplus_\lambda M_n(X_\lambda)$ isometrically (and by iteration, completely isometrically). Thus if $x \in M_n(\oplus_\lambda X_\lambda)$, then we have $\|x\|_n = \sup_\lambda \|x_\lambda\|_{M_n(X_\lambda)}$. Clearly the canonical inclusion and projection maps between $\oplus_\lambda X_\lambda$ and its 'λth summand' are complete isometries and complete quotient maps respectively. If X_λ are C^*-algebras then this direct sum is the usual C^*-algebra direct sum. If the X_λ are W^*-algebras then this direct sum is a W^*-algebra too, and is easy to work with in terms of the canonical central projections corresponding to the summands.

The ∞-direct sum has the following universal property. If Z is an operator space and $u_\lambda \colon Z \to X_\lambda$ are completely contractive linear maps, then there is a canonical complete contraction $Z \to \oplus_\lambda X_\lambda$ taking $z \in Z$ to the tuple $(u_\lambda(z))$.

If $X_\lambda = X$ for all $\lambda \in I$, then we usually write $\ell^\infty_I(X)$ for $\oplus_\lambda X_\lambda$. If $I = \mathbb{N}$ we may simply write $\ell^\infty(X)$. Note that we have

$$M_n(\ell^\infty_I(X)) \cong \ell^\infty_I(M_n(X)).$$

If $I = \mathbb{N}$ then one may define a c_0-direct sum operator space of operator spaces $X_1, X_2, \ldots$. This is simply the subspace of $\oplus^\infty_n X_n$ consisting of tuples (x_n) with $\lim_n \|x_n\| = 0$. We write $c_0(X)$ for this space if $X_n = X$ for all n.

1.2.18 (Operator valued continuous functions) Let Ω be a compact space, let X be an operator space, and consider the space $C(\Omega; X)$ of continuous X-valued functions on Ω (see A.3.2). This is an operator space with matrix norms coming from the identification $M_n(C(\Omega; X)) = C(\Omega; M_n(X))$. Clearly this 'canonical' operator space structure is given by the same formula as (1.3), and the natural embedding $C(\Omega; X) \subset \ell^\infty_\Omega(X)$ is a complete isometry. Note that if A is a C^*-algebra, then $C(\Omega; A)$ is a C^*-algebra, with product as pointwise multiplication and with $f^*(t) = (f(t))^*$ for any $f \in C(\Omega; A)$ and $t \in \Omega$.

Similarly if Ω is merely a locally compact space, then $C_0(\Omega; X)$ is an operator space as well, with $M_n(C_0(\Omega; X)) = C_0(\Omega; M_n(X))$ for all n.

1.2.19 (Mapping spaces) If X, Y are operator spaces, then the space $CB(X, Y)$ of completely bounded linear maps from X to Y, is also an operator space, with matrix norms determined via the canonical isomorphism between $M_n(CB(X, Y))$ and $CB(X, M_n(Y))$. Equivalently, if $[u_{ij}] \in M_n(CB(X, Y))$, then

$$\|[u_{ij}]\|_n = \sup\big\{\|[u_{ij}(x_{kl})]\|_{nm} : [x_{kl}] \in \mathrm{Ball}(M_m(X)),\ m \in \mathbb{N}\big\}. \tag{1.6}$$

Here the matrix $[u_{ij}(x_{kl})]$ is indexed on rows by i and k and on columns by j and l. Applying the above with n replaced by nN, to the space of matrices $M_N(M_n(CB(X, Y))) = M_{nN}(CB(X, Y))$, yields

$$M_n(CB(X, Y)) \cong CB(X, M_n(Y)) \quad \text{completely isometrically.} \tag{1.7}$$

One may see that (1.6) defines an operator space structure on $CB(X, Y)$ by appealing to Ruan's theorem 1.2.13 (directly or in the form of 1.2.16). Alternatively, one may see it as follows. Consider the set $I = \cup_n \mathrm{Ball}(M_n(X))$, and for

$x \in \text{Ball}(M_m(X)) \subset I$ set $n_x = m$. Consider the operator space direct sum (see 1.2.17) $\oplus_{x \in I}^{\infty} M_{n_x}(Y)$. Then the map from $CB(X,Y)$ to $\oplus_{x \in I} M_{n_x}(Y)$ taking u to the tuple $((I_{n_x} \otimes u)(x))_x \in \oplus_x M_{n_x}(Y)$ is (almost tautologically) a complete isometry. Thus $CB(X,Y)$ is an operator space.

1.2.20 (The dual of an operator space) The special case when $Y = \mathbb{C}$ in 1.2.19 is particularly important. In this case, for any operator space X, we obtain by 1.2.19 an operator space structure on $X^* = CB(X, \mathbb{C})$. The latter space equals $B(X, \mathbb{C})$ isometrically by 1.2.6. We call X^*, viewed as an operator space in this way, the *operator space dual* of X. This duality will be studied further in Sections 1.4–1.6. By (1.7) we have

$$M_n(X^*) \cong CB(X, M_n) \quad \text{completely isometrically.} \tag{1.8}$$

Note that the map implementing this isomorphism is exactly the canonical map (described in A.3.1) from $M_n \otimes X^*$ to $B(X, M_n)$.

1.2.21 (Minimal operator spaces) Let E be a Banach space, and consider the canonical isometric inclusion of E in the commutative C^*-algebra $C(\text{Ball}(E^*))$. Here E^* is equipped with the w^*-topology. This inclusion induces, via 1.2.3, an operator space structure on E, which is denoted by $\text{Min}(E)$. By (1.3), the resulting matrix norms on E are given by

$$\|[x_{ij}]\|_n = \sup\big\{\|[\varphi(x_{ij})]\| \,:\, \varphi \in \text{Ball}(E^*)\big\} \tag{1.9}$$

for $[x_{ij}] \in M_n(E)$. Thus every Banach space may be canonically considered to be an operator space. Since $\text{Min}(E) \subset C(\text{Ball}(E^*))$, we see from 1.2.6 that for any bounded linear u from an operator space Y into E, we have

$$\|u\colon Y \longrightarrow \text{Min}(E)\|_{\text{cb}} \;=\; \|u\colon Y \longrightarrow E\|. \tag{1.10}$$

From this last fact one easily sees that $\text{Min}(E)$ is the smallest operator space structure on E. Also, if Ω is any compact space and if $i\colon E \to C(\Omega)$ is an isometry, then the matrix norms inherited by E from the operator space structure of $C(\Omega)$, coincide again with those in (1.9). This may be seen by applying 1.2.6 to i and i^{-1}. Summarizing: 'minimal operator spaces' are exactly the operator spaces completely isometrically isomorphic to a subspace of a $C(K)$-space.

According to A.3.1, another way of stating (1.9) is to say that

$$M_n\big(\text{Min}(E)\big) = M_n \check{\otimes} E \tag{1.11}$$

isometrically via the canonical isomorphism.

1.2.22 (Maximal operator spaces) If E is a Banach space then $\text{Max}(E)$ is the largest operator space structure we can put on E. We define the matrix norms on $\text{Max}(E)$ by the following formula

$$\|[x_{ij}]\|_n = \sup\big\{\|[u(x_{ij})]\| \,:\, u \in \text{Ball}(B(E,Y)),\ \text{all operator spaces } Y\big\}.$$

This may be seen to be an operator space structure on X by using 1.2.16 say; and from this formula it is also clear that it is the largest such. Since every Banach space is isometric to an operator space (see 1.2.21), $\|\cdot\|_1$ is evidently the usual norm on E. It is clear from this formula that $\mathrm{Max}(E)$ has the property that for any operator space Y, and for any bounded linear $u\colon E \to Y$, we have

$$\|u\colon \mathrm{Max}(E) \longrightarrow Y\|_{\mathrm{cb}} \;=\; \|u\colon E \longrightarrow Y\|. \tag{1.12}$$

1.2.23 (Hilbert column and row spaces) If H is a Hilbert space then there are two canonical operator space structures on H most commonly considered. The first is the *Hilbert column space* H^c. Informally one should think of H^c as a 'column in $B(H)$'. Thus if $H = \ell^2_n$ then $H^c = M_{n,1}$, thought of as the matrices in M_n which are 'zero except on the first column'. We write this operator space also as C_n, and the 'row' version as R_n. For a general Hilbert space H there are several simple ways of describing H^c more precisely. For example, one may identify H^c with the concrete operator space $B(\mathbb{C}, H)$. Another equivalent description is as follows (we leave the equivalence as an exercise). If η is a fixed unit vector in H, then the set $H \otimes \eta$ of rank one operators $\zeta \otimes \eta$ is a closed subspace of $B(H)$ which is isometric to H via the map $\zeta \mapsto \zeta \otimes \eta$. (By convention, $\zeta \otimes \eta$ maps $\xi \in H$ to $\langle \xi, \eta\rangle \zeta$.) Thus we may transfer the operator space structure on $H \otimes \eta$ inherited from $B(H)$ over to H. The resulting operator space structure is independent of η and coincides with H^c. Indeed from the C^*-identity in $M_n(B(H))$ applied to $\|[\zeta_{ij} \otimes \eta]\|$, one immediately obtains

$$\|[\zeta_{ij}]\|_{M_n(H^c)} = \Big\| \Big[\sum_{k=1}^n \langle \zeta_{kj}, \zeta_{ki}\rangle\Big] \Big\|^{\frac{1}{2}}, \qquad [\zeta_{ij}] \in M_n(H). \tag{1.13}$$

If $T \in B(H, K)$ then $\|T\| = \|T\|_{\mathrm{cb}}$, where the latter is the norm taken in $CB(H^c, K^c)$. Indeed let $[\zeta_{ij}] \in M_n(H^c)$, and let $\alpha \in B(\ell^2_n, \ell^2_n(H))$ correspond to this matrix via the identity $M_n(H^c) = M_n(B(\mathbb{C}, H)) = B(\ell^2_n, \ell^2_n(H))$. Likewise let $\beta \in B(\ell^2_n, \ell^2_n(K))$ corresponding to $[T\zeta_{ij}]$. Then $\beta = (I_{\ell^2_n} \otimes T) \circ \alpha$, and hence $\|\beta\| \leq \|T\|\|\alpha\|$. This shows that $\|T\|_{\mathrm{cb}} \leq \|T\|$. More generally, we have

$$B(H, K) = CB(H^c, K^c) \quad \text{completely isometrically} \tag{1.14}$$

We give a quick proof of this identity in the Notes for this section.

A subspace K of a Hilbert column space H^c is again a Hilbert column space, as may be seen by considering (1.13). Similarly the quotient H^c/K^c is a Hilbert column space completely isometric to $(H \ominus K)^c$, as may be seen by applying 1.2.15 to the canonical (completely contractive by (1.14)) projection P from H^c onto $(H \ominus K)^c$.

We define *Hilbert row space* similarly. Recalling that $H^* \cong \bar{H}$ is a Hilbert space too, we identify H^r with the concrete operator space $B(\bar{H}, \mathbb{C})$. Analogues of the above results for H^c hold, except that there is a slight twist in the corresponding version of (1.14). Namely, although $B(H, K) = CB(H^r, K^r)$ isometrically, this is not true *completely isometrically.* Instead there is a canonical completely isometric isomorphism $B(H, K) \cong CB(\bar{K}^r, \bar{H}^r)$. We have

$$(H^c)^* \;\cong\; \bar{H}^r \quad \text{and} \quad (H^r)^* \;\cong\; \bar{H}^c \tag{1.15}$$

completely isometrically using the operator space dual structure in 1.2.20. The first relation is obtained by setting $K = \mathbb{C}$ in (1.14). Similarly, the second relation follows e.g. from the line above (1.15).

We write C and R for ℓ^2 with its column and row operator space structures respectively.

1.2.24 (The operator space $R \cap C$) We let $R \cap C$ be ℓ^2 with the operator space structure defined by the embedding $\ell^2 \hookrightarrow R \oplus^\infty C$ which takes any $x \in \ell^2$ to the pair (x, x). Let $(e_k)_{k\geq 1}$ denote the canonical basis of ℓ^2. Then it follows from (1.4) that for any $N \geq 1$ and any $x_1, \ldots, x_n$ in M_N, we have

$$\Big\|\sum_k x_k \otimes e_k\Big\|_{M_N(R\cap C)} \;=\; \max\Big\{\Big\|\sum_k x_k^* x_k\Big\|^{\frac{1}{2}}, \Big\|\sum_k x_k x_k^*\Big\|^{\frac{1}{2}}\Big\}. \tag{1.16}$$

We also note that if X is an operator space and $u\colon X \to \ell^2$ is a bounded linear map, then u is completely bounded from X into $R \cap C$ if and only if it is both completely bounded from X into R and from X into C. Moreover we have

$$\|u\colon X \to R \cap C\|_{\mathrm{cb}} \;=\; \max\big\{\|u\colon X \to R\|_{\mathrm{cb}}, \|u\colon X \to C\|_{\mathrm{cb}}\big\}. \tag{1.17}$$

1.2.25 (Opposite and adjoint) If X is an operator space, in $B(K, H)$ say, then we define the *adjoint operator space* to be the space $X^\star = \{x^* : x \in X\}$ (see 1.1.2). As an abstract operator space $X^\star$ is independent of the particular representation of X on H and K. Indeed we can alternatively define $X^\star$ as the set of formal symbols x^* for $x \in X$, with scalar product $\lambda x^* = (\bar{\lambda} x)^*$, and with matrix norms $\|[x_{ij}^*]\|_n = \|[x_{ji}]\|_n$, where the latter norm is taken in $M_n(X)$. The adjoint operator space is sometimes denoted by $\overline{X}$ by some authors. However we warn the reader that $X^\star$ is not the same as the conjugate operator space considered in [337].

If X is an operator space then we define the *opposite operator space* X^{op} to be the Banach space X with the 'transposed matrix norms' $\|[x_{ij}]\|_n^{\mathrm{op}} = \|[x_{ji}]\|_n$. Note that if A is a C^*-algebra, then these matrix norms on A^{op} coincide with the canonical matrix norms on the C^*-algebra which is A with its reversed multiplication. If X is a subspace of a C^*-algebra A, then X^{op} may be identified completely isometrically with the associated subspace of the C^*-algebra A^{op}.

If $u\colon X \to Y$, then we write u^{op} for u considered as a map from X^{op} to Y^{op}, and $u^\star$ for the map from $X^\star$ to $Y^\star$ defined by $u^\star(x^*) = u(x)^*$. These maps are completely bounded, completely contractive, completely isometric, etc., if u has these properties. There is a 'conjugate linear complete isometry' from X^{op} to $X^\star$, namely the map $x \mapsto x^*$.

1.2.26 (Matrix spaces) If X is an operator space, and I, J are cardinal numbers or sets, then we write $\mathbb{M}_{I,J}(X)$ for the set of $I \times J$ matrices whose finite submatrices have uniformly bounded norm. Such a matrix is normed by the supremum of the norms of its finite submatrices. Similarly there is an obvious way to define a norm on $M_n(\mathbb{M}_{I,J}(X))$ by equating this space with $\mathbb{M}_{I,J}(M_n(X))$, and one has $M_n(\mathbb{M}_I(X)) \cong \mathbb{M}_{n.I}(X)$, for $n \in \mathbb{N}$.

We are being deliberately careless here, and indeed in the rest of the book we often abusively blur the distinction between cardinals and sets. Technically if I, J are cardinals, we should fix sets I_0 and J_0 of cardinality I and J respectively, consider matrices $[x_{ij}]$ indexed by $i \in I_0$ and $j \in J_0$, and write $\mathbb{M}_{I_0,J_0}(X)$ instead of $\mathbb{M}_{I,J}(X)$. However if one chooses different sets I_1 and J_1 of these cardinalities, then there is an obvious completely isometric isomorphism $\mathbb{M}_{I_0,J_0}(X) \cong \mathbb{M}_{I_1,J_1}(X)$, so that with a little care our convention should not lead us into trouble. Or we may protect ourselves by fixing one well-ordered set associated with each cardinal.

We write $\mathbb{M}_I(X) = \mathbb{M}_{I,I}(X), C_I^w(X) = \mathbb{M}_{I,1}(X)$, and $R_I^w(X) = \mathbb{M}_{1,I}(X)$. If $I = \aleph_0$ we simply denote these spaces by $\mathbb{M}(X)$, $C^w(X)$ and $R^w(X)$ respectively. Also, $\mathbb{M}_{I,J}^{\text{fin}}(X)$ will denote the vector subspace of $\mathbb{M}_{I,J}(X)$ consisting of 'finitely supported matrices', that is, those matrices with only a finite number of nonzero entries. We write $\mathbb{K}_{I,J}(X)$ for the norm closure in $\mathbb{M}_{I,J}(X)$ of $\mathbb{M}_{I,J}^{\text{fin}}(X)$. We set $\mathbb{K}_I(X) = \mathbb{K}_{I,I}(X), C_I(X) = \mathbb{K}_{I,1}(X)$, and $R_I(X) = \mathbb{K}_{1,I}(X)$. Again we merely write $\mathbb{K}(X)$, $R(X)$ and $C(X)$ for these spaces if $I = \aleph_0$. If $X = \mathbb{C}$ then $C_I^w(\mathbb{C}) = C_I(\mathbb{C}) = (\ell_I^2)^c$ (see 1.2.23 for this notation), and we usually write this column Hilbert space as C_I. Similarly, $R_I = R_I(\mathbb{C}) = (\ell_I^2)^r$. We write $\mathbb{K}_{I,J}$ for $\mathbb{K}_{I,J}(\mathbb{C})$, and $\mathbb{M}_{I,J}$ for $\mathbb{M}_{I,J}(\mathbb{C})$.

It is fairly obvious that if $u \colon X \to Y$ is completely bounded, then so is the obvious *amplification* $u_{I,J} \colon \mathbb{M}_{I,J}(X) \to \mathbb{M}_{I,J}(Y)$, and $\|u_{I,J}\|_{\text{cb}} = \|u\|_{\text{cb}}$. Clearly $u_{I,J}$ also restricts to a completely bounded map from $\mathbb{K}_{I,J}(X)$ to $\mathbb{K}_{I,J}(Y)$. If u is a complete isometry, then so is $u_{I,J}$. Thus the $\mathbb{M}_{I,J}(\cdot)$ and $\mathbb{K}_{I,J}(\cdot)$ constructions are 'injective' in some sense.

For cardinals I, J, we leave it as an exercise that $\mathbb{M}_{I,J} \cong B(\ell_J^2, \ell_I^2)$. Via this identification, $\mathbb{K}_{I,J} = S^\infty(\ell_J^2, \ell_I^2)$. Thus for any Hilbert spaces K, H we have that $B(K, H) \cong \mathbb{M}_{I_0,J_0}$ for some cardinals I_0, J_0. We leave it as another exercise that

$$\mathbb{M}_{I,J}(\mathbb{M}_{I_0,J_0}) \cong \mathbb{M}_{I \times I_0, J \times J_0} \tag{1.18}$$

completely isometrically. Putting these two exercises together, we have established that for any cardinals I, J, we have

$$\mathbb{M}_{I,J}(B(K,H)) \cong B(K^{(J)}, H^{(I)}) \quad \text{completely isometrically.} \tag{1.19}$$

If X is an operator space then so is $\mathbb{M}_{I,J}(X)$. This may be seen by choosing a completely isometric embedding $X \subset B(H)$, and noting that by the 'injectivity' mentioned a few paragraphs back, and formula (1.19), we have $\mathbb{M}_{I,J}(X) \subset \mathbb{M}_{I,J}(B(H)) \cong B(H^{(J)}, H^{(I)})$ completely isometrically. If X is complete then so is $\mathbb{M}_{I,J}(X)$, since it is clearly norm closed in $\mathbb{M}_{I,J}(B(K,H))$.

For any operator space X, we have

$$\mathbb{M}_{I,J}(X) = C_I^w(R_J^w(X)) = R_J^w(C_I^w(X)). \tag{1.20}$$

One way to see this is to first check (1.20) in the case $X = B(H)$ using (1.19), and then use this fact to do the general case. By a similar argument, $\mathbb{M}_{I,J}(\mathbb{M}_{I_0,J_0}(X)) \cong \mathbb{M}_{I\times I_0,J\times J_0}(X)$ for any operator space X, generalizing (1.18).

1.2.27 (Infinite sums) Suppose that X, Y are subspaces of a (complete) operator algebra or C^*-algebra $A \subset B(H)$. Let I be an infinite set. If $x \in R_I^w(X)$ and $y \in C_I(Y)$, then the 'product' xy (defined to be $\sum_i x_i y_i$ if x and y have ith entries x_i and y_i respectively) actually converges in norm to an element of A. To see this, we use the following notation. If z is an element of $R_I^w(X)$ or $C_I(Y)$, and if $\Delta \subset I$, write z_Δ for z but with all entries outside Δ 'switched to zero'. Since $y \in C_I(Y)$, given $\epsilon > 0$ there is a finite set $\Delta \subset I$, such that $\|y - y_\Delta\| = \|y_{\Delta^c}\| < \epsilon$. If Δ' is a finite subset of I not intersecting Δ then

$$\Big\|\sum_{i\in\Delta'} x_i y_i\Big\| = \|x_{\Delta'} y_{\Delta'}\| \leq \|x_{\Delta'}\|\|y_{\Delta'}\| \leq \|x\|\|y_{\Delta'}\| < \|x\|\epsilon.$$

Hence the sum converges in norm as claimed. Thus we have

$$R_I^w(X)\, C_I(Y) \subset XY \quad \text{and} \quad R_I(X)\, C_I^w(Y) \subset XY, \tag{1.21}$$

where XY is as defined in 1.1.2, a closed subset of A. Also $\|xy\| \leq \|x\|\|y\|$ for x, y as above, as may be seen from a computation identical to the first part of the second last centered equation.

Proposition 1.2.28 *For any operator space X and cardinal I, we have that $CB(C_I, X) \cong R_I^w(X)$ and $CB(R_I, X) \cong C_I^w(X)$ completely isometrically.*

Proof We prove just the first relation. Define $L\colon R_I^w(X) \to CB(C_I, X)$ by $L(x)(z) = \sum_i x_i z_i$, for $x \in R_I^w(X), z \in C_I$. This map is well defined, by the argument for (1.21) for example. It is also easy to check, by looking at the partial sums of this series as in (1.21), that L is contractive. Conversely, for u in $CB(C_I, X)$, let $x \in R_I^w(X)$ have ith entry $u(e_i)$, where (e_i) is the canonical basis. It is not hard to see that $\|x\|_{R_I^w(X)} \leq \|u\|_{\mathrm{cb}}$, and $L(x) = u$. Thus L is a surjective isometry. This together with (1.7) yields

$$M_m(CB(C_I, X)) \cong CB(C_I, M_m(X)) \cong R_I^w(M_m(X)) \cong M_m(R_I^w(X))$$

isometrically. From this one sees that L is a complete isometry. $\square$

Proposition 1.2.29 *If X and Y are operator spaces then there are canonical complete isometries*

$$\mathbb{K}_{I,J}(CB(X,Y)) \hookrightarrow CB(X, \mathbb{K}_{I,J}(Y)) \hookrightarrow CB(X, \mathbb{M}_{I,J}(Y)) \cong \mathbb{M}_{I,J}(CB(X,Y)).$$

In particular, if $Y = \mathbb{C}$, we have $CB(X, \mathbb{M}_{I,J}) \cong \mathbb{M}_{I,J}(X^*)$.

Proof Since $\mathbb{K}_{I,J}(Y) \subset \mathbb{M}_{I,J}(Y)$, the middle inclusion is evident. There is a canonical map $\Theta\colon \mathbb{M}_{I,J}(CB(X,Y)) \to CB(X, \mathbb{M}_{I,J}(Y))$, which takes an element

$[u_{ij}]$ from $\mathbb{M}_{I,J}(CB(X,Y))$, to the map $x \mapsto [u_{ij}(x)]$. Also there is a canonical map $CB(X, \mathbb{M}_{I,J}(Y)) \to \mathbb{M}_{I,J}(CB(X,Y))$, which takes u to the matrix $[\pi_{ij} \circ u]$, where π_{ij} is the projection of $\mathbb{M}_{I,J}(Y)$ onto its i-j entry. It is rather easy to check that these maps are mutual inverses, and are both completely contractive. Hence they are complete isometries. Thus $\mathbb{M}_{I,J}(CB(X,Y)) \cong CB(X, \mathbb{M}_{I,J}(Y))$. Finally, the isometry above taking $\mathbb{M}_{I,J}(CB(X,Y))$ into $CB(X, \mathbb{M}_{I,J}(Y))$, clearly takes $\mathbb{M}^{\mathrm{fin}}_{I,J}(CB(X,Y))$ into $CB(X, \mathbb{K}_{I,J}(Y))$. By density, $\mathbb{K}_{I,J}(CB(X,Y))$ embeds completely isometrically in $CB(X, \mathbb{K}_{I,J}(Y))$. $\square$

1.2.30 (Interpolation) We recall the complex interpolation method for Banach spaces (e.g. see [33, Chapter 4]). Suppose that (X_0, X_1) is a *compatible couple* of Banach spaces. This means that we are given a topological vector space $\mathcal{Z}$, and one-to-one continuous linear mappings from X_0 to $\mathcal{Z}$ and X_1 to $\mathcal{Z}$. Regard X_0 and X_1 as subspaces of $\mathcal{Z}$. Their 'sum' $X_0 + X_1 \subset \mathcal{Z}$ is, by definition, the space of all $x_0 + x_1$, with $x_0 \in X_0$ and $x_1 \in X_1$. This is a Banach space with norm

$$\|x\| = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \,:\, x_0 \in X_0,\ x_1 \in X_1,\ x = x_0 + x_1\}.$$

We let $\mathcal{S}$ denote the strip of all complex numbers z with $0 \leq \mathrm{Re}(z) \leq 1$ and we let $\mathcal{F} = \mathcal{F}(X_0, X_1)$ be the space of all bounded and continuous functions $f : \mathcal{S} \to X_0 + X_1$ such that the restriction of f to the interior of $\mathcal{S}$ is analytic, and such that the maps $t \mapsto f(it)$ and $t \mapsto f(1+it)$ belong to $C_0(\mathbb{R}; X_0)$ and $C_0(\mathbb{R}; X_1)$ respectively. Then $\mathcal{F}$ is a Banach space for the norm

$$\|f\|_{\mathcal{F}} = \max\bigl\{\sup_t \|f(it)\|_{X_0}\,,\, \sup_t \|f(1+it)\|_{X_1}\,\bigr\}. \tag{1.22}$$

For any $0 \leq \theta \leq 1$, the *interpolation space* $X_\theta = [X_0, X_1]_\theta$ is the subspace of $X_0 + X_1$ formed by all x such that $x = f(\theta)$ for some $f \in \mathcal{F}$. This turns out to be a Banach space for the norm $\|x\|_{X_\theta} = \inf\bigl\{\|f\|_{\mathcal{F}} \,:\, f \in \mathcal{F},\ f(\theta) = x\bigr\}$. If we let $\mathcal{F}_\theta = \mathcal{F}_\theta(X_0, X_1)$ be the subspace of all $f \in \mathcal{F}$ for which $f(\theta) = 0$, we see that the mapping $f \mapsto f(\theta)$ induces an isometric isomorphism

$$X_\theta = \mathcal{F}(X_0, X_1)/\mathcal{F}_\theta(X_0, X_1). \tag{1.23}$$

Assume now that X_0 and X_1 are operator spaces. Then each interpolation space X_θ has a 'natural' operator space structure. Indeed note from (1.22) that the mapping which takes any $f \in \mathcal{F}(X_0, X_1)$ to the pair of its restrictions to the lines $\{\mathrm{Re}(z) = 0\}$ and $\{\mathrm{Re}(z) = 1\}$, induces an isometric embedding

$$\mathcal{F}(X_0, X_1) \subset C_0(\mathbb{R}; X_0) \oplus^\infty C_0(\mathbb{R}; X_1).$$

By 1.2.17 and 1.2.18, we may consider $\mathcal{F}(X_0, X_1)$ as an operator space, the norm on $M_n(\mathcal{F}(X_0, X_1))$ being inherited from $C_0(\mathbb{R}; M_n(X_0)) \oplus^\infty C_0(\mathbb{R}; M_n(X_1))$.

Then taking the resulting quotient operator space structure on $\mathcal{F}/\mathcal{F}_\theta$ and applying (1.23), makes X_θ an operator space. More explicitly, the matrix norms on the operator space X_θ are given for any $[x_{jk}] \in M_n(X_\theta)$ by

$$\|[x_{jk}]\|_n = \inf\Big\{\max\big\{\sup_t \|[f_{jk}(it)]\|_{M_n(X_0)}\,,\, \sup_t \|[f_{jk}(1+it)]\|_{M_n(X_1)}\,\big\}\Big\},$$

the infimum taken over all $f_{jk} \in \mathcal{F}(X_0, X_1)$ such that $f_{jk}(\theta) = x_{jk}$, for all j, k.

Observe that for each $n \geq 1$, we have natural one-to-one continuous linear maps $M_n(X_0) \to M_n(\mathcal{Z})$ and $M_n(X_1) \to M_n(\mathcal{Z})$. Hence $(M_n(X_0), M_n(X_1))$ is a compatible couple of Banach spaces. It follows easily from the above discussion that $M_n(\mathcal{F}(X_0, X_1)) = \mathcal{F}(M_n(X_0), M_n(X_1))$ isometrically, and hence

$$M_n(X_\theta) = \big[M_n(X_0), M_n(X_1)\big]_\theta. \tag{1.24}$$

This formula readily implies that a key interpolation theorem (see [33, Theorem 4.1.2]) extends to completely bounded maps. Namely, let (Y_0, Y_1) be another compatible couple of operator spaces, and let $u\colon X_0 + X_1 \to Y_0 + Y_1$ be a linear map. If u is completely bounded as a map from X_0 into Y_0, and from X_1 into Y_1, then u is completely bounded from X_θ into Y_θ, for any $\theta \in (0,1)$, with

$$\|u\colon X_\theta \longrightarrow Y_\theta\|_{\mathrm{cb}} \,\leq\, \|u\colon X_0 \longrightarrow Y_0\|_{\mathrm{cb}}^{1-\theta}\, \|u\colon X_1 \longrightarrow Y_1\|_{\mathrm{cb}}^{\theta}.$$

1.2.31 (Ultraproducts) Let $\mathcal{U}$ be an ultrafilter on a set I and let $(X_i)_{i\in I}$ be a family of operator spaces. We let $\mathcal{N}_{\mathcal{U}} \subset \oplus_i^\infty X_i$ be the space of all $(x_i)_i$ such that $\lim_{\mathcal{U}} \|x_i\|_{X_i} = 0$. By definition, the ultraproduct of the family $(X_i)_{i\in I}$ along $\mathcal{U}$ is the quotient operator space

$$\prod_{i\in I} X_i/\mathcal{U} = \big(\oplus_i^\infty X_i\big)/\mathcal{N}_{\mathcal{U}}$$

from 1.2.17 and 1.2.14. It is easy to check that for any $x = (x_i)_i$ in $\oplus_i M_n(X_i)$, the norm of its class $\dot{x}$ modulo $M_n(\mathcal{N}_{\mathcal{U}})$ is equal to

$$\|\dot{x}\| = \lim_{\mathcal{U}} \|x_i\|_{M_n(X_i)}.$$

This implies the 'injectivity' of ultraproducts. Namely if $(Y_i)_{i\in I}$ is another family of operator spaces such that $X_i \subset Y_i$ completely isometrically for each $i \in I$, then $\prod_{i\in I} X_i/\mathcal{U} \subset \prod_{i\in I} Y_i/\mathcal{U}$ completely isometrically.

Finally, we observe that if $(A_i)_{i\in I}$ is a family of C^*-algebras, then $\mathcal{N}_{\mathcal{U}}$ is an ideal of $\oplus_i^\infty X_i$ and hence their ultraproduct is fairly clearly a C^*-algebra. Note that if $(a_i)_i$ and $(b_i)_i$ belong to $\oplus_i^\infty A_i$, the product $\dot{a}\dot{b}$ of their classes modulo $\mathcal{N}_{\mathcal{U}}$ is the class of $(a_i b_i)_i$.

1.3 COMPLETELY POSITIVE MAPS

1.3.1 (Unital operator spaces) Recall that a C^*-algebra A is called *unital* if it contains an identity element 1. We say that an operator space X is *unital* if it has a distinguished element usually written as e or 1, called the *identity of* X, such that there exists a complete isometry $u\colon X \to A$ into a unital C^*-algebra with $u(e) = 1$. A *unital-subspace* of such X is a subspace containing e.

1.3.2 (Operator systems) An *operator system* is a unital-subspace $\mathcal{S}$ of a unital C^*-algebra A which is *selfadjoint*, that is, $x^* \in \mathcal{S}$ if and only if $x \in \mathcal{S}$. A *subsystem* of an operator system $\mathcal{S}$ is a selfadjoint linear subspace of $\mathcal{S}$ containing the 'identity' 1 of $\mathcal{S}$. If $\mathcal{S}$ is an operator system, a subsystem of a C^*-algebra A, then $\mathcal{S}$ has a distinguished 'positive cone' $\mathcal{S}_+ = \{x \in \mathcal{S} : x \geq 0 \text{ in } A\}$. We also write $\mathcal{S}_{sa}$ for the real vector space of selfadjoint elements x (i.e. those satisfying $x = x^*$) in $\mathcal{S}$. Then $\mathcal{S}$ has an associated ordering $\leq$, namely we say that $x \leq y$ if x, y are selfadjoint and $y - x \in \mathcal{S}_+$. By the usual trick, any element of an operator system $\mathcal{S}$ is of the form $h + ik$ for $h, k \in \mathcal{S}_{sa}$. Also, if $h \in \mathcal{S}_{sa}$ then $\|h\|1 + h$ and $\|h\|1 - h$ are positive. Thus $\mathcal{S}_{sa} = \mathcal{S}_+ - \mathcal{S}_+$.

A linear map $u : \mathcal{S} \to \mathcal{S}'$ between operator systems is called $*$-*linear* if $u(x^*) = u(x)^*$ for all $x \in \mathcal{S}$. Some authors say that such a map is *selfadjoint*. We say that u is *positive* if $u(\mathcal{S}_+) \subset \mathcal{S}'_+$. By facts at the end of the last paragraph it is easy to see that a positive map is $*$-linear. The operator system $M_n(\mathcal{S})$, which is a subsystem of $M_n(A)$, has a 'positive cone' too, and thus it makes sense to talk about *completely positive maps* between operator systems. These are the maps u such that $u_n = I_{M_n} \otimes u\colon M_n(\mathcal{S}) \to M_n(\mathcal{S}')$ is positive for all $n \in \mathbb{N}$. Indeed the morphisms in the category of operator systems are often taken to be the unital completely positive maps.

Suppose that $\mathcal{S}$ is a subsystem of a unital C^*-algebra. By the Hahn–Banach theorem and A.4.2 (resp. (A.11)), it follows that $\mathcal{S}_{sa}$ (resp. $\mathcal{S}_+$) is exactly the set of elements $x \in \mathcal{S}$ such that $\varphi(x) \in \mathbb{R}$ (resp. $\varphi(x) \geq 0$) for all $\varphi \in (\mathrm{Span}\{1, x\})^*$ with $\varphi(1) = \|\varphi\| = 1$. From this it is clear that if $u\colon \mathcal{S}_1 \to \mathcal{S}_2$ is a contractive unital linear map between operator systems, then u is a positive and $*$-linear map. Applying this principle to u_n, we see that a completely contractive unital linear map between operator systems is completely positive.

Clearly an isomorphism between operator systems which is unital and completely positive, and has completely positive inverse, preserves all the 'order'. Such a map is called a *complete order isomorphism*. The range of a completely positive unital map between operator systems is clearly also an operator system; we say that such a map is a *complete order injection* if it is a complete order isomorphism onto its range.

The following simple fact relates the norm to the matrix order, and is an elementary exercise using the definition of a positive operator. Namely, if x is an element of a unital C^*-algebra or operator system A, or if $x \in B(K, H)$, then

$$\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad \|x\| \leq 1. \tag{1.25}$$

Here '≥ 0' means 'positive in $M_2(A)$' (or 'positive in $B(H \oplus K)$').

1.3.3 It is easy to see from (1.25) that a completely positive unital map u between operator systems is completely contractive. (For example, to see that u is contractive, take $\|x\| \leq 1$, and apply u_2 to the associated positive matrix in (1.25). This is positive, so that using (1.25) again we see that $\|u(x)\| \leq 1$.) Putting this together with some facts from 1.3.2 we see that a unital map between operator systems is completely positive if and only if it is completely contractive; and in this case the map is $*$-linear. If, further, u is one-to-one, then by applying the above to u and u^{-1} one sees immediately that a unital map between operator systems is a complete order injection if and only if it is a complete isometry.

We omit the well-known proofs of the following two results.

Theorem 1.3.4 (Stinespring) *Let A be a unital C^*-algebra. A linear map $u\colon A \to B(H)$ is completely positive if and only if there is a Hilbert space K, a unital $*$-homomorphism $\pi\colon A \to B(K)$, and a bounded linear $V\colon H \to K$ such that $u(a) = V^*\pi(a)V$ for all $a \in A$. This can be accomplished with $\|u\|_{\mathrm{cb}} = \|V\|^2$. Also, this equals $\|u\|$. If u is unital then we may take V to be an isometry; in this case we may view $H \subset K$, and we have $u(\cdot) = P_H\pi(\cdot)_{|H}$.*

Theorem 1.3.5 (Arveson's extension theorem) *If $\mathcal{S}$ is a subsystem of a unital C^*-algebra A, and if $u\colon \mathcal{S} \to B(H)$ is completely positive, then there exists a completely positive map $\hat{u}\colon A \to B(H)$ extending u.*

Indeed, if u is unital, then 1.3.5 may be easily seen from 1.2.10 and 1.3.3 (although usually one *proves* 1.2.10 using the completely positive variant).

Lemma 1.3.6 (Arveson) *Suppose that X is a unital-subspace of an operator system, and suppose that $u\colon X \to B(K)$ is a unital contraction (resp. complete contraction, complete isometry) with range Y. Then there exists a positive map (resp. completely positive map, complete order isomorphism) $\tilde{u}$ between the operator systems $X + X^\star$ and $Y + Y^\star$, which extends u, namely the map $x_1 + x_2^* \mapsto u(x_1) + u(x_2)^*$, for $x_1, x_2 \in X$.*

Proof Note that the 'contraction' result here applied to the amplifications u_n will imply the 'complete contraction' result; and the complete isometry case will then follow by considering u^{-1}.

Suppose that u is a contraction, and consider u restricted to the operator subsystem $X \cap X^\star$. This is a unital contraction and therefore is positive and $*$-linear by a fact in 1.3.2. From this it is easy to check directly that the formula above for $\tilde{u}$ is well defined.

Suppose that $x_1 + x_2^* \in X + X^\star$ is positive, and that ζ is a unit vector in K, so that $\varphi = \langle \cdot\zeta, \zeta\rangle$ is a state on $B(K)$. Then $\varphi \circ u$ extends by the Hahn–Banach theorem to a contractive unital functional ψ on $X + X^\star$. By the aforementioned fact from 1.3.2, ψ is therefore positive and $*$-linear. Thus

$$\langle (u(x_1) + u(x_2)^*)\zeta, \zeta\rangle = \psi(x_1) + \overline{\psi(x_2)} = \psi(x_1 + x_2^*) \geq 0.$$

Hence $u(x_1) + u(x_2)^* \geq 0$. $\square$

1.3.7 (The diagonal) Because of this last result, if X is a unital operator space (see 1.3.1) then there is an essentially unique operator system, written as $X + X^\star$, which is spanned by X and its adjoint space $X^\star$. Indeed, if $u\colon X \to B(H)$ is any unital complete isometry into $B(H)$ (or into an operator system), then by 1.3.6 the operator system $u(X) + u(X)^\star$ is (up to unital complete order isomorphism) independent of the particular u. We usually identify two unital operator spaces up to unital completely isometric isomorphism.

By the same principle, any such X contains a canonical operator system, namely $\Delta(X) = \{x \in X : u(x)^* \in u(X)\}$, where u is any unital complete isometry as in the last paragraph. This is well defined independently of u. We call $\Delta(X)$ the *diagonal* of X.

The following follows immediately from a fact in 1.3.2 and the last definition:

Corollary 1.3.8 *If $\mathcal{S}$ is an operator system, if Y is a unital operator space, and if $u\colon \mathcal{S} \to Y$ is a unital contraction, then* $\mathrm{Ran}(u) \subset \Delta(Y)$*, and u is positive.*

Proposition 1.3.9 (A Kadison–Schwarz inequality) *If $u\colon A \to B$ is a unital completely positive (or equivalently unital completely contractive) linear map between unital C^*-algebras, then $u(a)^* u(a) \leq u(a^* a)$, for all $a \in A$.*

Proof By 1.3.4 we have $u = V^* \pi(\cdot) V$, with $\|V\| \leq 1$ and π a $*$-homomorphism. Thus $u(a)^* u(a) = V^* \pi(a)^* V V^* \pi(a) V \leq V^* \pi(a)^* \pi(a) V = u(a^* a)$. □

Corollary 1.3.10 *Let $u\colon A \to B$ be a completely isometric unital surjection between unital C^*-algebras. Then u is a $*$-isomorphism.*

Proof By 1.3.9 applied to both u and u^{-1} we have $u(x)^* u(x) = x^* x$ for all $x \in A$. Now use the polarization identity (see (1.1)). □

Proposition 1.3.11 *Let $u\colon A \to B$ be as in* 1.3.9. *Suppose that $c \in A$, and that c satisfies $u(c)^* u(c) = u(c^* c)$. Then $u(ac) = u(a) u(c)$ for all $a \in A$.*

Proof Suppose that $B \subset B(H)$. We write $u = V^* \pi(\cdot) V$ as in Stinespring's theorem, with $V^* V = I_H$. Let $P = V V^*$ be the projection onto $V(H)$. By hypothesis $V^* \pi(c)^* P \pi(c) V = V^* \pi(c)^* \pi(c) V$. For $\zeta \in H$, set $\eta = \pi(c) V \zeta$. Then $\|P\eta\|^2 = \langle V^* \pi(c)^* P \pi(c) V \zeta, \zeta \rangle = \|\eta\|^2$. Thus $P\eta = \eta$, and $V V^* \pi(c) V = \pi(c) V$. Therefore $u(a) u(c) = V^* \pi(a) V V^* \pi(c) V = V^* \pi(a) \pi(c) V = u(ac)$. □

1.3.12 (Completely positive bimodule maps) An immediate consequence of 1.3.11: Suppose that $u\colon A \to B$ is as in 1.3.9, and that there is a C^*-subalgebra C of A with $1_A \in C$, such that $\pi = u_{|C}$ is a $*$-homomorphism. Then

$$u(ac) = u(a)\pi(c) \quad \text{and} \quad u(ca) = \pi(c) u(a) \qquad (a \in A, c \in C).$$

Theorem 1.3.13 (Choi and Effros) *Suppose that A is a unital C^*-algebra, and that $\Phi\colon A \to A$ is a unital, completely positive (or equivalently by* 1.3.3*, completely contractive), idempotent map. Then we may conclude:*

(1) *$R = \mathrm{Ran}(\Phi)$ is a C^*-algebra with respect to the original norm, involution, and vector space structure, but new product $r_1 \circ_\Phi r_2 = \Phi(r_1 r_2)$.*

(2) $\Phi(ar) = \Phi(\Phi(a)r)$ *and* $\Phi(ra) = \Phi(r\Phi(a))$, *for* $r \in R$ *and* $a \in A$.

(3) *If* B *is the* C^**-subalgebra of* A *generated by the set* R, *and if* R *is given the product* $\circ_\Phi$, *then* $\Phi_{|B}$ *is a* $*$*-homomorphism from* B *onto* R.

Proof (2) By linearity and the fact that a positive map is $*$-linear (see 1.3.2), we may assume that a, r are selfadjoint. Set

$$d \;=\; d^* \;=\; \begin{bmatrix} 0 & r \\ r^* & a \end{bmatrix}.$$

Then $\Phi_2(d^2) \geq (\Phi_2(d))^2$ by the Kadison–Schwarz inequality 1.3.9, so that

$$\begin{bmatrix} \Phi(r^2) & \Phi(ra) \\ \Phi(ar) & * \end{bmatrix} \;\geq\; \begin{bmatrix} r^2 & r\Phi(a) \\ \Phi(a)r & * \end{bmatrix}.$$

Here $*$ is used for a term we do not care about. Applying Φ_2 gives

$$\begin{bmatrix} \Phi(r^2) & \Phi(ra) \\ \Phi(ar) & * \end{bmatrix} \;\geq\; \begin{bmatrix} \Phi(r^2) & \Phi(r\Phi(a)) \\ \Phi(\Phi(a)r) & * \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 0 & \Phi(ra) - \Phi(r\Phi(a)) \\ \Phi(ar) - \Phi(\Phi(a)r) & * \end{bmatrix} \;\geq\; 0,$$

which implies that $\Phi(ra) - \Phi(r\Phi(a)) = 0$ and $\Phi(ar) - \Phi(\Phi(a)r) = 0$.

(1) By (2) we have for $r_1, r_2, r_3 \in R$ that

$$(r_1 \circ_\Phi r_2) \circ_\Phi r_3 \;=\; \Phi(\Phi(r_1 r_2) r_3) \;=\; \Phi(r_1 r_2 r_3).$$

Similarly, $r_1 \circ_\Phi (r_2 \circ_\Phi r_3) = \Phi(r_1 r_2 r_3)$, which shows that the multiplication is associative. It is easy to check that R (with original norm, involution, and vector space structure, but new multiplication) satisfies the conditions necessary to be a C^*-algebra. For example:

$$(r_1 \circ_\Phi r_2)^* \;=\; \Phi(r_1 r_2)^* \;=\; \Phi(r_2^* r_1^*) \;=\; r_2^* \circ_\Phi r_1^*.$$

We check the C^*-identity using the Kadison–Schwarz inequality 1.3.9:

$$\|r^* \circ_\Phi r\| \;=\; \|\Phi(r^* r)\| \;\geq\; \|\Phi(r)^* \Phi(r)\| = \|r^* r\| = \|r\|^2,$$

and conversely,

$$\|r\|^2 \;=\; \|r^* r\| \;\geq\; \|\Phi(r^* r)\| \;=\; \|r^* \circ_\Phi r\|.$$

(3) This will follow if we can prove that $\Phi(r_1 r_2 \cdots r_n) = r_1 \circ_\Phi r_2 \cdots \circ_\Phi r_n$, for $r_i \in R$. This follows in turn by induction on n. Supposing that it is true for $n = k$, we see that $r_1 \circ_\Phi r_2 \cdots \circ_\Phi r_{k+1}$ equals

$$\Phi((r_1 \circ_\Phi r_2 \cdots \circ_\Phi r_k) r_{k+1}) = \Phi(\Phi(r_1 r_2 \cdots r_k) r_{k+1}) = \Phi(r_1 r_2 \cdots r_k r_{k+1}),$$

using (2) in the last equality. $\square$

It is important to note, and easy to check, that the canonical matrix norms from 1.2.3 for the C^*-algebra $\Phi(A)$ in the result above, coincide with its canonical matrix norms as a subspace of A. This may be seen by the uniqueness of a complete C^*-norm on a $*$-algebra (which in turn is immediate from A.5.8), and an application of Theorem 1.3.13 to the canonically associated projection Φ_n on $M_n(A)$, for each $n \in \mathbb{N}$.

1.3.14 (The Paulsen system) If X is a subspace of $B(H)$, we define the *Paulsen system* to be the operator system

$$\mathcal{S}(X) \;=\; \begin{bmatrix} \mathbb{C}I_H & X \\ X^\star & \mathbb{C}I_H \end{bmatrix} \;=\; \left\{ \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \;:\; x, y \in X,\; \lambda,\, \mu \in \mathbb{C} \right\}$$

in $M_2(B(H))$, where the entries λ and μ in the last matrix stand for λI_H and μI_H respectively. The following important lemma shows that as an operator system (i.e. up to complete order isomorphism) $\mathcal{S}(X)$ only depends on the operator space structure of X, and not on its representation on H.

Lemma 1.3.15 (Paulsen) *Suppose that for $i = 1, 2$, we are given Hilbert spaces H_i, K_i, and linear subspaces $X_i \subset B(K_i, H_i)$. Suppose that $u \colon X_1 \to X_2$ is a linear map. Let $\mathcal{S}_i$ be the following operator system inside $B(H_i \oplus K_i)$:*

$$\mathcal{S}_i = \begin{bmatrix} \mathbb{C}I_{H_i} & X_i \\ X_i^\star & \mathbb{C}I_{K_i} \end{bmatrix} .$$

If u is contractive (resp. completely contractive, completely isometric), then

$$\Theta : \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \;\mapsto\; \begin{bmatrix} \lambda & u(x) \\ u(y)^* & \mu \end{bmatrix}$$

is positive (resp. completely positive and completely contractive, a complete order injection) as a map from $\mathcal{S}_1$ to $\mathcal{S}_2$.

Proof Suppose that z is a positive element of $\mathcal{S}_1$. Thus

$$z \;=\; \begin{bmatrix} a & x \\ x^* & b \end{bmatrix}$$

where a and b are positive. Since $z \geq 0$ if and only if $z + \epsilon 1 \geq 0$ for all $\epsilon > 0$, we may assume that a and b are invertible. Then

$$\begin{bmatrix} a^{-\frac{1}{2}} & 0 \\ 0 & b^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} a & x \\ x^* & b \end{bmatrix} \begin{bmatrix} a^{-\frac{1}{2}} & 0 \\ 0 & b^{-\frac{1}{2}} \end{bmatrix} \;=\; \begin{bmatrix} 1 & a^{-\frac{1}{2}} x b^{-\frac{1}{2}} \\ b^{-\frac{1}{2}} x^* a^{-\frac{1}{2}} & 1 \end{bmatrix} \;\geq\; 0.$$

Hence by (1.25), we have that $\|a^{-\frac{1}{2}} x b^{-\frac{1}{2}}\| \leq 1$. Applying u we obtain that $\|a^{-\frac{1}{2}} u(x) b^{-\frac{1}{2}}\| \leq 1$. Reversing the argument above now shows that $\Theta(z) \geq 0$. So Θ is positive, and a similar argument shows that it is completely positive if u is completely contractive. By 1.3.3 we have that Θ is completely contractive in that case. If in addition u is a complete isometry, then applying the above to u and u^{-1} we obtain the final assertion. □

1.4 OPERATOR SPACE DUALITY

An operator space Y is said to be a *dual operator space* if Y is completely isometrically isomorphic to the operator space dual (see 1.2.20) X^* of an operator space X. We also say that X is an *operator space predual* of Y, and sometimes we write X as Y_*. If X, Y are dual operator spaces then we write $w^*CB(X, Y)$ for the space of w^*-continuous completely bounded maps from X to Y.

Unless otherwise indicated, in what follows the symbol X^* denotes the dual space *together with its dual operator space structure* as defined in 1.2.20. Of course X^{**} is considered as the dual operator space of X^*.

Proposition 1.4.1 *If X is an operator space then $X \subset X^{**}$ completely isometrically via the canonical map i_X.*

Proof Let $X \subset B(H)$. By the definitions, i_X is completely contractive. To see that i_X is completely isometric, it suffices to find for a given $n \in \mathbb{N}$, $\epsilon > 0$, and $[x_{kl}] \in M_n(X)$, an integer m and a completely contractive $u \colon B(H) \to M_m$ such that $\|[u(x_{kl})]\| \geq \|[x_{kl}]\| - \epsilon$. For such $[x_{kl}]$ and ϵ, by (1.2) we may choose $\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n \in H$ with $\sum_l \|\zeta_l\|^2 = \sum_k \|\eta_k\|^2 = 1$ and

$$\Big|\sum_{k,l} \langle x_{kl}\zeta_l, \eta_k\rangle\Big| \geq \|[x_{kl}]\| - \epsilon.$$

Let $K = \mathrm{Span}\,\{\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n\}$, and let $u \colon B(H) \to B(K)$ be the map $u(x) = P_K x_{|K}$. If $m = \dim(K)$, then we may view u as mapping into M_m. It is completely contractive, by the remark after 1.2.8 for example. Finally,

$$\|[u(x_{kl})]\| \geq \Big|\sum_{k,l} \langle u(x_{kl})\zeta_l, \eta_k\rangle\Big| = \Big|\sum_{k,l} \langle x_{kl}\zeta_l, \eta_k\rangle\Big| \geq \|[x_{kl}]\| - \epsilon,$$

which is the desired inequality. □

1.4.2 (Remarks) From 1.4.1 we have for any $[x_{ij}] \in M_n(X)$ that

$$\|[x_{ij}]\|_n = \sup\{\|[\varphi_{kl}(x_{ij})]\| : m \in \mathbb{N},\ [\varphi_{kl}] \in \mathrm{Ball}(M_m(X^*))\} \tag{1.26}$$

In other words, the following canonical isomorphism is a complete isometry:

$$M_n(X) \cong w^*CB(X^*, M_n) \subset CB(X^*, M_n). \tag{1.27}$$

Another consequence of 1.4.1, is that if X is an operator space which as a Banach space is reflexive, then $X \cong X^{**}$ completely isometrically.

1.4.3 (The adjoint map) The 'adjoint' or 'dual' u^* of a completely bounded map $u \colon X \to Y$ between operator spaces is completely bounded from Y^* to X^*, with $\|u^*\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$, as may be seen from the obvious computation. Indeed using (1.26) during this computation or applying 1.4.1, one sees that $\|u\|_{\mathrm{cb}} = \|u^*\|_{\mathrm{cb}}$.

Direct computations from the definitions also show that if u is a complete quotient map then u^* is a complete isometry. It is slightly harder to see that

u is completely isometric if and only if u^* is a complete quotient map. Indeed, if u is a complete isometry, then we may regard $X \subset Y$, and then an element in the open ball of $M_n(X^*) \cong CB(X, M_n)$ may be 'extended' by 1.2.10 to an element in the open ball of $CB(Y, M_n) = M_n(Y^*)$. This shows that u^* is a complete quotient map. Conversely, if u^* is a complete quotient map then u^{**} is a complete isometry, so that u is a complete isometry (using 1.4.1). Finally, one may see as in the Banach space case, that for complete operator spaces, if u^* is a complete isometry then u is a complete quotient map (e.g. see A.2.3 in [149]).

Thus u is a complete isometry if and only if u^{**} is a complete isometry.

1.4.4 (Duality of subspaces and quotients) The operator space versions of the usual Banach duality of subspaces and quotients follow easily from 1.4.3. If X is a subspace of Y, then applying 1.4.3 to the inclusion map $X \hookrightarrow Y$ yields the fact that $X^* \cong Y^*/X^\perp$ completely isometrically via the canonical map. Similarly, the dual of the canonical quotient map $Y \to Y/X$ is the canonical complete isometry $(Y/X)^* \cong X^\perp$. The predual versions go through too with the same proofs as in the Banach space case: if X is a w^*-closed subspace of a dual operator space Y, then $(Y_*/X_\perp)^* \cong (X_\perp)^\perp = X$ as dual operator spaces. Also, $(X_\perp)^* \cong Y/(X_\perp)^\perp = Y/X$ completely isometrically.

1.4.5 (The trace class operator space) If H is a Hilbert space then $B(H)$ is a dual operator space. More precisely, let us equip its predual Banach space $S^1(H)$ (e.g. see A.1.2) with the operator space structure it inherits from $B(H)^*$ via the canonical isometric inclusion $S^1(H) \hookrightarrow B(H)^*$. Then $B(H) = S^1(H)^*$ completely isometrically. Indeed the canonical map from $B(H)$ to $S^1(H)^*$ is completely contractive by definition. That this map is completely isometric follows from the fact, included in the proof of 1.4.1, that for any $n \in \mathbb{N}$, $\epsilon > 0$, and $[x_{kl}] \in M_n(B(H))$, we can find an integer m and a w^*-continuous completely contractive $u\colon B(H) \to M_m$ such that $\|[u(x_{kl})]\| \geq \|[x_{kl}]\| - \epsilon$.

Similarly, $B(K, H)$ is the dual operator space of the space $S^1(H, K)$ of trace class operators, the latter regarded as a subspace of $B(K, H)^*$. Henceforth, when we write $S^1(H, K)$ we will mean the operator space predual of $B(K, H)$ described above. Similarly, we will henceforth also view $S^1_n = M^*_n$ as an operator space.

Lemma 1.4.6 *Any w^*-closed subspace X of $B(H)$ is a dual operator space. Indeed, if $Y = S^1(H)/X_\perp$ is equipped with its quotient operator space structure inherited from $S^1(H)$, then $X \cong Y^*$ completely isometrically.*

Proof This follows from 1.4.4 and 1.4.5. □

In particular this shows that any W^*-algebra equipped with its 'natural' operator space structure (see 1.2.3) is a dual operator space.

The converse of 1.4.6 is true too, as we see next, so that 'dual operator spaces', and the w^*-closed subspaces of some $B(H)$, are essentially the same thing.

Lemma 1.4.7 *Any dual operator space is completely isometrically isomorphic, via a homeomorphism for the w^*-topologies, to a w^*-closed subspace of $B(H)$, for some Hilbert space H.*

Proof Suppose that W is a dual operator space, with predual X. Let $Y = \mathbb{C}$, and recall from 1.2.19 the construction of a complete isometry

$$W \;=\; CB(X,Y) \;\longrightarrow\; \oplus^{\infty}_{x\in I}\, M_{n_x}(Y) \;=\; \oplus_{x\in I}\, M_{n_x},$$

namely the map J taking $w \in W$ to the tuple $([\langle w, x_{ij}\rangle])_x$ in $\oplus_x M_{n_x}$. Since the maps $w \mapsto \langle w, x_{ij}\rangle$ are w^*-continuous, and since $\oplus^{\text{fin}}_x M^*_{n_x}$ is dense in the Banach space predual $\oplus^1_x M^*_{n_x}$ of $\oplus_x M_{n_x}$, it is easy to see that J is w^*-continuous too. Thus by A.2.5, W is completely isometrically and w^*-homeomorphically isomorphic to a w^*-closed subspace of $\oplus_x M_{n_x}$. If the latter is regarded as a von Neumann subalgebra of $B(H)$ say, then W is completely isometrically and w^*-homeomorphically isomorphic to a w^*-closed subspace of $B(H)$. $\square$

1.4.8 (W^*-continuous extensions) If X and Y are two operator spaces and if $u\colon X \to Y^*$ is completely bounded, then its (unique) w^*-continuous extension $\tilde{u}\colon X^{**} \to Y^*$ provided by A.2.2 is completely bounded, with $\|\tilde{u}\|_{\text{cb}} = \|u\|_{\text{cb}}$. Indeed recall from A.2.2 that $\tilde{u} = i^*_Y \circ u^{**}$; and this extension clearly satisfies the asserted norm equality (using the first paragraph in 1.4.3). Note that since $\tilde{u}$ is w^*-continuous, we have $\tilde{u}(X^{**}) \subset \overline{u(X)}^{w*}$. The above also shows that

$$CB(X, Y^*) = w^*CB(X^{**}, Y^*) \tag{1.28}$$

isometrically via the mapping $u \mapsto \tilde{u}$. In fact (1.28) can easily be made a complete isometry with the help of the later item 1.6.2.

By 1.4.5, the last paragraph applies in particular to $B(K,H)$ valued maps.

1.4.9 (The second dual) Let X be an operator space, and fix $n \in \mathbb{N}$. We wish to compare the spaces $M_n(X^{**})$ (equipped with its 'operator space dual' matrix norms as in 1.2.20), and $M_n(X)^{**}$. First note that they can be canonically identified as topological vector spaces, as may $M_n(X^*)$ and $M_n(X)^*$ (this is just the simple fact that if $E = F \oplus \cdots \oplus F$ is a finite direct sum of copies of a Banach space F, which has been assigned a norm compatible with the norm on F, then E^* is canonically algebraically and topologically isomorphic to $F^* \oplus \cdots \oplus F^*$, the latter with any norm compatible with the norm on F^*). We will prove in 1.4.11 below that this identification is an isometry. As a first easy step, let us check that the identity mapping from $M_n(X)^{**}$ to $M_n(X^{**})$ implementing this identification, is a contraction. For this purpose, let $\eta = [\eta_{ij}] \in M_n \otimes X^{**}$, and assume that its norm in $M_n(X)^{**}$ is less than or equal to 1. By Goldstine's lemma A.2.1, there is a net $(x^s)_s$ in $\text{Ball}(M_n(X))$ such that $x^s \to \eta$ in the w^*-topology of $M_n(X)^{**}$. Let $\varphi = [\varphi_{pq}]$ be an element of $\text{Ball}(M_m(X^*))$ for some $m \geq 1$. The fact that $x^s \to \eta$ in the w^*-topology of $M_n(X)^{**}$, is equivalent to the fact that $x^s_{ij} \to \eta_{ij}$ in the w^*-topology of X^{**} for all $1 \leq i, j \leq n$. Hence we deduce that

$$[\langle \eta_{ij}, \varphi_{pq}\rangle] \;=\; \lim_s\, [\langle \varphi_{pq}, x^s_{ij}\rangle].$$

By (1.6) or (1.26), the norm of the latter matrix is dominated by 1. Thus $\|[\langle \eta_{ij}, \varphi_{pq}\rangle]\| \leq 1$. By (1.6) again, we deduce that $\|[\eta_{ij}]\|_{M_n(X^{**})} \leq 1$, which proves the result.

1.4.10 (The second dual of a C^*-algebra) If A is a C^*-algebra, then the second dual A^{**} has two canonical operator space structures. The first is its 'operator space dual' matrix norms (see 1.2.20); when we write $M_n(A^{**})$ in the lines below, we will be using these norms. The second are those from 1.2.3, arising from the fact that A^{**} is a C^*-algebra (see A.5.6). We claim that these two operator space structures are the same. To see this we will need to use notation and facts from A.5.6. In particular we let $\pi_u : A \to B(H_u)$ denote the universal representation of A, and we write $A^{\dagger\dagger}$ for the W^*-algebra $\widetilde{\pi_u}(A^{**})$ (see the proof of A.5.6). The claim will follow if we can prove for any fixed $n \geq 1$ that

$$M_n(A)^{**} \cong M_n(A^{**}) \cong M_n(A^{\dagger\dagger}) \quad \text{isometrically} \tag{1.29}$$

via the canonical maps. The first of these maps is the contraction from $M_n(A)^{**}$ to $M_n(A^{**})$ discussed in 1.4.9. The second map in (1.29) is $I_{M_n} \otimes \widetilde{\pi_u}$, which is a contraction since according to 1.4.8, the mapping $\widetilde{\pi_u}$ is a complete contraction. To establish (1.29), we need only prove that the resulting contraction $\rho : M_n(A)^{**} \longrightarrow M_n(A^{\dagger\dagger})$ is isometric. It is clearly one-to-one. We regard $M_n(A)^{**}$ as a C^*-algebra by applying A.5.6 to $M_n(A)$. It therefore suffices to check that ρ is a $*$-homomorphism. Regarding ρ as valued in $B(H_u^{(n)})$, we have $\langle \rho(\eta)\,\zeta, \xi \rangle = \sum_{i,j} \langle \widetilde{\pi_u}(\eta_{ij})\zeta_j, \xi_i \rangle$, for $\zeta = [\zeta_i], \xi = [\xi_i] \in H_u^{(n)}$, and $\eta = [\eta_{ij}]$ as in 1.4.9. Since $\widetilde{\pi_u}$, and the maps $\eta \mapsto \eta_{ij}$, are w^*-continuous, it follows that ρ is w^*-continuous too. By the w^*-continuity properties of the involution and product in a W^*-algebra (see A.5.1), it suffices to prove that the restriction of ρ to $M_n(A)$ is a $*$-homomorphism. Since the latter equals $I_{M_n} \otimes \pi_u$, we are done.

The last result has many consequences. For example, we can use it to see that $S^\infty(H)^* = S^1(H)$ completely isometrically, complementing the observation in 1.4.5. Also we obtain:

Theorem 1.4.11 *If X is an operator space then $M_{m,n}(X)^{**} \cong M_{m,n}(X^{**})$ completely isometrically for all $m, n \in \mathbb{N}$ (via an isomorphism extending the identity map on $M_{m,n}(X)$).*

Proof First suppose that $m = n$, and choose a C^*-algebra A with $X \subset A$ completely isometrically. Then $X^{**} \subset A^{**}$ completely isometrically by 1.4.3, hence we have both $M_n(X)^{**} \subset M_n(A)^{**}$, and $M_n(X^{**}) \subset M_n(A^{**})$, isometrically. Under the identifications between $M_n(A)^{**}$ and $M_n(A^{**})$ and between $M_n(X)^{**}$ and $M_n(X^{**})$ discussed above, these two embeddings are easily seen to be the same. Hence the isometry $M_n(A^{**}) = M_n(A)^{**}$ provided by 1.4.10, implies that we also have $M_n(X^{**}) = M_n(X)^{**}$ isometrically. The complete isometry follows by iterating the isometric case.

The case $n \neq m$ may be derived from the above, viewing $M_{m,n}(X)$ as a subspace of $M_k(X)$ where $k = \max\{m, n\}$. We leave this as an exercise. $\square$

1.4.12 (Duality of Min and Max) For any Banach space E, we have

$$\mathrm{Min}(E)^* = \mathrm{Max}(E^*) \qquad \text{and} \qquad \mathrm{Max}(E)^* = \mathrm{Min}(E^*). \tag{1.30}$$

To see this, note that by using (1.8), (1.11) and (1.12), and the basic properties of $\check{\otimes}$ seen in A.3.1, we have

$$M_n(\mathrm{Max}(E)^*) \cong CB(\mathrm{Max}(E), M_n) = B(E, M_n) \cong M_n \check{\otimes} E^* \cong M_n(\mathrm{Min}(E^*)).$$

That is, $\mathrm{Max}(E)^* = \mathrm{Min}(E^*)$. Therefore $\mathrm{Max}(E^*)^* = \mathrm{Min}(E^{**})$. However we claim that $\mathrm{Min}(E^{**}) = \mathrm{Min}(E)^{**}$. This claim may be seen using the fact that 'minimal operator spaces' are completely isometric to subspaces of unital commutative C^*-algebras (i.e. of $C(K)$-spaces), the fact that the second dual of a complete isometry is a complete isometry (see 1.4.4), and 1.4.10. Hence $\mathrm{Max}(E^*)$ and $\mathrm{Min}(E)^*$ are two operator space structures on E^* with the same operator space dual, and therefore they are completely isometric, by 1.4.1.

1.4.13 (The 1-direct sum) For a family $\{X_\lambda : \lambda \in I\}$ of operator spaces, let $\oplus_\lambda^{\mathrm{fin}} X_\lambda$ be the set of 'finitely supported' elements of the algebraic direct sum of the X_λ. There is a canonical one-to-one map $\mu\colon \oplus_\lambda^{\mathrm{fin}} X_\lambda \to (\oplus_\lambda^\infty X_\lambda^*)^*$, and we may define the 1-direct sum $\oplus_\lambda^1 X_\lambda$ by identifying it with the closure of $\mu(\oplus_\lambda^{\mathrm{fin}} X_\lambda)$ in the dual operator space $(\oplus_\lambda^\infty X_\lambda^*)^*$. Evidently, the norm on $\oplus_\lambda^1 X_\lambda$ is exactly the usual 'ℓ^1-norm' $\sum_\lambda \|x_\lambda\|$; and a tuple (x_λ) is in $\oplus_\lambda^1 X_\lambda$ if and only if $x_\lambda \in X$ for all λ, and $\sum_\lambda \|x_\lambda\| < \infty$. If $X_\lambda = X$ for all $\lambda \in I$ then we write $\ell_I^1(X)$ for $\oplus_\lambda^1 X$.

It is easy to argue that the canonical inclusion and projection maps ϵ_λ and π_λ between $\oplus_\lambda^1 X_\lambda$ and its 'λth summand' are complete isometries and complete quotient maps respectively. Also,

$$(\oplus_\lambda^1 X_\lambda)^* \;\cong\; \oplus_\lambda^\infty X_\lambda^* \quad \text{as dual operator spaces.} \tag{1.31}$$

Similarly, the dual of an ∞-direct sum of finitely many operator spaces (or the dual of a c_0-direct sum of infinitely many) is completely isometric to the 1-direct sum of the duals. We leave these assertions to the reader to check. In particular,

$$\ell_I^1(X)^* = \ell_I^\infty(X^*) \qquad \text{and} \qquad c_0(X)^* = \ell^1(X^*).$$

The 1-direct sum has the following useful universal property. Namely, if Z is another operator space and $u_\lambda\colon X_\lambda \to Z$ are completely contractive linear maps, then there is a canonical complete contraction $u\colon \oplus_\lambda^1 X_\lambda \to Z$ such that $u \circ \epsilon_\lambda = u_\lambda$. One may see this with a diagram chase: u_λ^* maps Z^* to X_λ^*, so that by the universal property of the ∞-direct sum we obtain a map $v\colon Z^* \to \oplus_\lambda X_\lambda^*$. Then $v^*\colon (\oplus_\lambda X_\lambda^*)^* \to Z^{**}$. Composing v with the canonical map $\oplus_\lambda^1 X_\lambda \to (\oplus_\lambda X_\lambda^*)^*$ gives a map $u\colon \oplus_\lambda^1 X_\lambda \to Z^{**}$. One easily checks that $u \circ \epsilon_\lambda = u_\lambda$, whence u actually maps into Z, and we obtain the desired result. This universal property may be rephrased as an isometric isomorphism

$$CB(\oplus_i^1 X_i, Z) \;\cong\; \oplus_i^\infty CB(X_i, Z).$$

Replacing Z by $M_n(Z)$ and using (1.7) shows that this relation also holds completely isometrically.

The canonical inclusion and projection maps ϵ_λ and π_λ between $\oplus^\infty_\lambda X^*_\lambda$ and its 'λth summand' are w^*-continuous. This may be seen by dualizing the inclusion and projection maps between $\oplus^1_\lambda X_\lambda$ and its summands.

Any operator space is a complete quotient of a 1-sum of spaces of the form $S^1_n = M^*_n$. This may be seen, for example, by applying the second last fact in 1.4.3 to the map J in the proof of 1.4.7.

Another frequently used property of these direct sums is as follows. Suppose that $u_\lambda\colon X_\lambda \to Y_\lambda$ is a complete contraction for each $\lambda \in I$. Then this family of maps clearly induces a single contraction $u = (u_\lambda)\colon\ \oplus^1_\lambda X_\lambda \to \oplus^1_\lambda Y_\lambda$. It also clearly induces a complete contraction $(u_\lambda)\colon\ \oplus^\infty_\lambda X_\lambda \to \oplus^\infty_\lambda Y_\lambda$; which is completely isometric if every u_λ is. Applying the first fact in the last sentence to the family of maps u^*_λ, gives a complete contraction from $\oplus^\infty_\lambda Y^*_\lambda$ to $\oplus^\infty_\lambda X^*_\lambda$, and it is easy to see from the norm density of the 'finitely supported' tuples in a 1-direct sum, that this map is the dual of the map u above, and hence is w^*-continuous. From this and 1.4.3 we deduce that u is a complete contraction. We have also shown that families (v_λ) of w^*-continuous complete contractions between dual spaces canonically induce a single w^*-continuous complete contraction between the ∞-direct sums of the spaces.

1.5 OPERATOR SPACE TENSOR PRODUCTS

As we said in the introduction, the reader should feel free to skim through these results, returning later when necessary for a definition or fact. In this section and the next we tend to leave more details than in previous sections as exercises for the interested reader.

1.5.1 (Minimal tensor product) Let X and Y be operator spaces, and let $X \otimes Y$ denote their algebraic tensor product. Any finite rank bounded map between operator spaces is 'composed' of scalar functionals, and hence is automatically completely bounded by 1.2.6. Thus the correspondences between tensor products and finite rank mappings discussed in A.3.1 yield embeddings $X \otimes Y \hookrightarrow CB(Y^*, X)$ and $X \otimes Y^* \hookrightarrow CB(Y, X)$. The *minimal tensor product* $X \otimes_{\min} Y$ may then be defined to be (the completion of) $X \otimes Y$ in the matrix norms inherited from the operator space structure on $CB(Y^*, X)$ described in 1.2.19. That is,

$$X \otimes_{\min} Y \hookrightarrow CB(Y^*, X) \quad \text{completely isometrically.} \tag{1.32}$$

That is, if $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$, then the norm of u in $X \otimes_{\min} Y$ equals

$$\sup \Big\| \Big[\sum_k x_k\ \psi_{ij}(y_k) \Big] \Big\|_{M_n(X)}, \tag{1.33}$$

the supremum taken over all finite matrices $[\psi_{ij}]$ of norm 1, where $\psi_{ij} \in Y^*$. A similar formula holds for $u \in M_n(X \otimes Y)$. From this similar form of (1.33) and (1.26), we see that the matrix norms on $X \otimes_{\min} Y$ are also given by the formula

$$\|[w_{rs}]\|_n \ = \ \sup\big\{ \|[(\varphi_{kl} \otimes \psi_{ij})(w_{rs})]\| \big\} \tag{1.34}$$

for $[w_{rs}] \in M_n(X \otimes Y)$, where the supremum is taken over all finite matrices $[\varphi_{kl}]$ and $[\psi_{ij}]$ of norm 1, where $\varphi_{kl} \in X^*$ and $\psi_{ij} \in Y^*$, and where $\varphi_{kl} \otimes \psi_{ij}$ denotes the obvious functional on $X \otimes Y$ formed from φ_{kl} and ψ_{ij}.

We see from (1.34) that $\otimes_{\min}$ is *commutative*, that is

$$X \otimes_{\min} Y \;=\; Y \otimes_{\min} X$$

as operator spaces. It is also easy to see from (1.34) that $\otimes_{\min}$ is *functorial*. That is, if X_i and Y_i are operator spaces for $i = 1, 2$, and if $u_i \colon X_i \to Y_i$ are completely bounded, then the map $x \otimes y \mapsto u_1(x) \otimes u_2(y)$ on $X_1 \otimes X_2$ has a unique continuous extension to a map $u_1 \otimes u_2 \colon X_1 \otimes_{\min} X_2 \to Y_1 \otimes_{\min} Y_2$, with $\|u_1 \otimes u_2\|_{\mathrm{cb}} \leq \|u_1\|_{\mathrm{cb}} \|u_2\|_{\mathrm{cb}}$. As an exercise, the reader could check that this is actually an equality, but we shall not need this. If, further, the u_i are completely isometric, then so is $u_1 \otimes u_2$. This latter fact is called the *injectivity* of the tensor product. To prove it, since $u_1 \otimes u_2 = (u_1 \otimes I) \circ (I \otimes u_2)$, we may by symmetry reduce the argument to the case that $Y_2 = X_2$, $u_2 = I_{X_2}$, $X_1 \subset Y_1$ and that u_1 is this inclusion map. Then the result we want follows from (1.32) and the obvious fact that $CB(X_2^*, X_1) \subset CB(X_2^*, Y_1)$ completely isometrically.

For any operator spaces X, Y, we have

$$X \otimes_{\min} Y^* \;\hookrightarrow\; CB(Y, X) \quad \text{completely isometrically.} \tag{1.35}$$

To prove this, note by the injectivity discussed above that we may assume that $X = B(H)$. However, $CB(Y, B(H)) = w^*CB(Y^{**}, B(H)) \subset CB(Y^{**}, B(H))$ isometrically by (1.28). Since the norm on $B(H) \otimes_{\min} Y^*$ is induced by the embedding of $B(H) \otimes Y^*$ in the latter space, $B(H) \otimes_{\min} Y^* \subset CB(Y, B(H))$ isometrically. The complete isometry follows by changing H into $H^{(n)}$, and using (1.2), (1.7), and the fact that $M_n(X \otimes_{\min} Y) = M_n(X) \otimes_{\min} Y$ for operator spaces X, Y (which in turn follows from (1.32) and (1.7)).

1.5.2 (Further properties of $\otimes_{\min}$) Suppose that H_1, H_2 are Hilbert spaces, and consider the canonical map $\pi \colon B(H_1) \otimes B(H_2) \to B(H_1 \otimes^2 H_2)$. This is the map taking a rank one tensor $S \otimes T$ in $B(H_1) \otimes B(H_2)$ to the map on $H_1 \otimes^2 H_2$ denoted also by $S \otimes T$ in 1.1.4. We claim that π actually is a complete isometry when $B(H_1) \otimes B(H_2)$ is given its norm as a subspace of $B(H_1) \otimes_{\min} B(H_2)$. To see this, we choose a cardinal I such that $H_2 = \ell^2_I$, so that we both have $\mathbb{M}_I \cong B(H_2)$ and $H_1 \otimes^2 H_2 \cong \ell^2_I(H_1)$. By (1.35) and 1.4.5, $B(H_1) \otimes_{\min} B(H_2) \hookrightarrow CB(S^1(H_1), \mathbb{M}_I)$. However, by 1.2.29, and (1.19), we have

$$CB(S^1(H_1), \mathbb{M}_I) \cong \mathbb{M}_I(B(H_1)) \cong B(H_1 \otimes^2 H_2).$$

This proves the claim.

Thus if X and Y are subspaces of $B(H_1)$ and $B(H_2)$ respectively, then by the injectivity of this tensor product, we have that $X \otimes_{\min} Y$ is completely isometrically isomorphic to the closure in $B(H_1 \otimes^2 H_2)$ of the span of the operators $x \otimes y$ on $H_1 \otimes^2 H_2$, for $x \in X, y \in Y$. We remark, in passing, that the above says that the minimal tensor product coincides with the tensor product of the same name used in C^*-algebra theory, or with the so-called *spatial tensor product.* Indeed note that if A and B are C^*-subalgebras of $B(H_1)$ and $B(H_2)$ respectively, then $A \otimes_{\min} B$ may be identified with the closure of a $*$-subalgebra of $B(H_1 \otimes^2 H_2)$. Thus $A \otimes_{\min} B$ is a C^*-algebra. If A and B are also commutative, then so is $A \otimes_{\min} B$, since it is the closure of a commutative $*$-subalgebra.

From the last paragraph and 1.2.2 it is clear that for any operator space X,

$$M_n \otimes_{\min} X \;\cong\; M_n(X) \tag{1.36}$$

completely isometrically. By similar reasoning, using also (1.19), we have

$$\mathbb{K}_{I,J} \otimes_{\min} X \;\cong\; \mathbb{K}_{I,J}(X). \tag{1.37}$$

Indeed, in the case $I = J$, both sides of (1.37) correspond to the closure of $\mathbb{M}_I^{\mathrm{fin}}(X)$ in $B(H^{(I)})$, if $X \subset B(H)$.

Similarly, it follows from the second last paragraph, and from the fact that $B((H_1 \otimes^2 H_2) \otimes^2 H_3) \cong B(H_1 \otimes^2 (H_2 \otimes^2 H_3))$, that $\otimes_{\min}$ is *associative.* That is,

$$(X_1 \otimes_{\min} X_2) \otimes_{\min} X_3 = X_1 \otimes_{\min} (X_2 \otimes_{\min} X_3). \tag{1.38}$$

Accordingly, this space will be merely denoted by $X_1 \otimes_{\min} X_2 \otimes_{\min} X_3$. Similarly, one may consider the N-fold minimal tensor product $X_1 \otimes_{\min} \cdots \otimes_{\min} X_N$ of any N-tuple of operator spaces.

Proposition 1.5.3 *Let E, F be Banach spaces and let X be an operator space.*

(1) $\mathrm{Min}(E) \otimes_{\min} X = E \check{\otimes} X$ *as Banach spaces.*

(2) $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F) = \mathrm{Min}(E \check{\otimes} F)$ *as operator spaces.*

(3) *For any compact space Ω we have (with notation as in* 1.2.18*),*

$$C(\Omega) \otimes_{\min} X = C(\Omega; X) \quad \textit{completely isometrically.} \tag{1.39}$$

Proof We have isometric embeddings $\mathrm{Min}(E) \otimes_{\min} X \subset CB(X^*, \mathrm{Min}(E))$ and $E \check{\otimes} X \subset B(X^*, E)$ by (1.32) and (A.1). However $CB(X^*, \mathrm{Min}(E)) = B(X^*, E)$ by (1.10), which proves (1). The isometry in (2) follows from (1). Thus the complete isometry in (2) will follow if $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F)$ is a minimal operator space. But this is clear if we use the 'injectivity' of $\otimes_{\min}$, the fact that 'minimal operator spaces' are the subspaces of $C(K)$-spaces, and the observation in 1.5.2 that the minimal tensor product of commutative C^*-algebras is a commutative C^*-algebra, and hence is a $C(K)$-space.

We now prove (3). Since $C(\Omega)$ is a minimal operator space, the relation $C(\Omega) \otimes_{\min} X = C(\Omega; X)$ holds isometrically by (1) and (A.2). By definition (see 1.2.18), $M_n(C(\Omega; X)) = C(\Omega; M_n(X))$. Replacing X by $M_n(X)$ in the previous relation, we deduce that $C(\Omega) \otimes_{\min} M_n(X) = M_n(C(\Omega; X))$ isometrically for all $n \in \mathbb{N}$. However by (1.36), and also by the associativity and commutativity of $\otimes_{\min}$, we have

$$C(\Omega) \otimes_{\min} M_n(X) \cong C(\Omega) \otimes_{\min} M_n \otimes_{\min} X \cong M_n\big(C(\Omega) \otimes_{\min} X\big).$$

This shows that $M_n(C(\Omega) \otimes_{\min} X) = M_n(C(\Omega; X))$ isometrically. $\square$

1.5.4 (Haagerup tensor product) Before we define this tensor product, we introduce an intimately related class of bilinear maps. Suppose that X, Y, and W are operator spaces, and that $u\colon X \times Y \to W$ is a bilinear map. For $n, p \in \mathbb{N}$, define a bilinear map $M_{n,p}(X) \times M_{p,n}(Y) \to M_n(W)$ by

$$(x, y) \longmapsto \Big[\sum_{k=1}^{p} u(x_{ik}, y_{kj})\Big]_{i,j},$$

where $x = [x_{ij}] \in M_{n,p}(X)$ and $y = [y_{ij}] \in M_{p,n}(Y)$. If $p = n$ we write this map as u_n. If the norms of these bilinear maps (as defined in A.3.3) are uniformly bounded over $p, n \in \mathbb{N}$, then we say that u is *completely bounded*, and write the supremum of these norms as $\|u\|_{\mathrm{cb}}$. It is easy to see (by adding rows and columns of zeroes to make $p = n$) that $\|u\|_{\mathrm{cb}} = \sup_n \|u_n\|$. We say that u is *completely contractive* if $\|u\|_{\mathrm{cb}} \leq 1$. Completely bounded multilinear maps of three variables have a similar definition (involving the expression $[\sum_{k,l} u(x_{ik}, y_{kl}, z_{lj})]$), and similarly for four or more variables. We remark that if $v\colon X \to B(H)$ and $w\colon Y \to B(H)$ are completely bounded linear maps, then it is easy to see that the bilinear map $(x, y) \mapsto v(x)w(y)$ is completely bounded in the sense above, and has completely bounded norm dominated by $\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}}$. Indeed this boils down to the fact that $B(H^{(n)})$ is a Banach algebra.

Let X, Y be operator spaces. For $n \in \mathbb{N}$ and $z \in M_n(X \otimes Y)$ we define

$$\|z\|_{\mathrm{h}} = \inf\{\|x\|\|y\|\}, \tag{1.40}$$

where the infimum is taken over all $p \in \mathbb{N}$, and all ways to write $z = x \odot y$, where $x \in M_{n,p}(X), y \in M_{p,n}(Y)$. Here $x \odot y$ denotes the formal matrix product of x and y using the $\otimes$ sign as multiplication: namely $x \odot y = [\sum_{k=1}^{p} x_{ik} \otimes y_{kj}]$. We leave it as an exercise (see 9.2.1 in [149] for more details if needed) that the expressions in (1.40) for all $n \in \mathbb{N}$, define an operator seminorm structure on $X \otimes Y$ in the sense of 1.2.16.

Now let $u\colon X \times Y \to W$ be a bilinear map which is completely contractive in the sense above. Let $\tilde{u}\colon X \otimes Y \to W$ be the canonically associated linear map. For $z \in M_n(X \otimes Y)$, we have by the definitions above that

$$\|\tilde{u}_n(z)\| \leq \|z\|_{\mathrm{h}}, \tag{1.41}$$

where the latter quantity is as defined in (1.40). If φ and ψ are contractive functionals on X and Y respectively, then using 1.2.6 and the fact at the end of the second last paragraph, we see that the bilinear map $(x, y) \mapsto \varphi(x)\psi(y)$ is completely contractive. Thus from (1.41) and the definition of the Banach space injective tensor norm of z (see A.3.1), we deduce that the latter norm of an element $z \in X \otimes Y$ is dominated by $\|z\|_{\mathrm{h}}$. Hence indeed $\|\cdot\|_{\mathrm{h}}$ is a norm. By the fact at the end of the last paragraph, together with Ruan's theorem, we see that the completion $X \otimes_h Y$ of $X \otimes Y$ with respect to $\|\cdot\|_{\mathrm{h}}$ is an operator space. This operator space is called the *Haagerup tensor product*. Note that the canonical bilinear map $\otimes\colon X \times Y \to X \otimes_h Y$ is completely contractive in the sense above.

Using (1.41) we see that if $u\colon X \times Y \to W$ is a bilinear map with associated linear map $\tilde{u}\colon X \otimes Y \to W$, then u is completely bounded if and only if $\tilde{u}$ extends to a completely bounded linear map on $X \otimes_h Y$. Moreover we have

$$\|u\|_{\mathrm{cb}} = \big\|\tilde{u}\colon X \otimes_h Y \longrightarrow W\big\|_{\mathrm{cb}}$$

in that case. The above property means that the Haagerup tensor product *linearizes completely bounded bilinear maps*. A moments thought shows that this is a *universal property*. That is, suppose that (W, μ) is a pair consisting of an operator space W, and a completely contractive bilinear map $\mu\colon X \times Y \to W$, such that the span of the range of μ is dense in W, and which possesses the following property:

> Given any operator space Z and given any completely bounded bilinear map $u\colon X \times Y \to Z$, then there exists a linear completely bounded $\tilde{u}\colon W \to Z$ such that $\tilde{u}(\mu(x, y)) = u(x, y)$ for all $x \in X, y \in Y$, and such that $\|\tilde{u}\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$.

Then $X \otimes_h Y \cong W$ via a complete isometry v satisfying $v \circ \otimes = \mu$.

1.5.5 (Further properties of the Haagerup tensor product) By the definition of the Haagerup tensor product, or by its universal property, it is easy to see that this tensor product is *functorial*. That is, if $u_i : X_i \to Y_i$ are completely bounded maps between operator spaces, then $u_1 \otimes u_2 : X_1 \otimes_h X_2 \to Y_1 \otimes_h Y_2$ is completely bounded, and $\|u_1 \otimes u_2\|_{\mathrm{cb}} \leq \|u_1\|_{\mathrm{cb}}\, \|u_2\|_{\mathrm{cb}}$.

It is also easy to show that the Haagerup tensor product is *associative* (e.g. see [318]). That is, $(X_1 \otimes_h X_2) \otimes_h X_3 \cong X_1 \otimes_h (X_2 \otimes_h X_3)$ completely isometrically. Accordingly, this space will be merely denoted by $X_1 \otimes_h X_2 \otimes_h X_3$. The induced norms on $M_n(X_1 \otimes X_2 \otimes X_3)$ may be described by the '3-variable' version of (1.40). From this one may see that $X_1 \otimes_h X_2 \otimes_h X_3$ has the universal property of 'linearizing' completely bounded trilinear maps (see discussion at the end of 1.5.4). Similar assertions clearly hold for the N-fold Haagerup tensor product $X_1 \otimes_h \cdots \otimes_h X_N$ of any N-tuple of operator spaces. Also, the Haagerup tensor product is *injective*. That is, if u_1 and u_2 in the last paragraph are completely isometric, then so is $u_1 \otimes u_2$. See the Notes for directions to a selection of proofs of this fact. It is much easier to see that the Haagerup tensor product is *projective*, that is, if u_1 and u_2 in the last paragraph are complete quotient maps, then so is $u_1 \otimes u_2$. To see this, note that by the functoriality, the map $u_1 \otimes u_2$ is a complete contraction. Let $z \in M_n(Y_1 \otimes Y_2)$, with $\|z\|_{\mathrm{h}} < 1$. By definition, we may write $z = y_1 \odot y_2$, where $y_1 \in M_{n,p}(Y_1)$, $y_2 \in M_{p,n}(Y_2)$ both have norm < 1. Then $y_1 = (u_1)_{n,p}(x_1)$ and $y_2 = (u_2)_{p,n}(x_2)$ for $x_1 \in M_{n,p}(X_1)$, $x_2 \in M_{p,n}(X_2)$, both of norm < 1. Let $w = x_1 \odot x_2 \in M_n(X_1 \otimes_h X_2)$, this matrix has norm < 1, and $(u_1 \otimes u_2)_n(w) = z$. By an obvious density argument, this shows that $u_1 \otimes u_2$ above is a complete quotient map.

The Haagerup tensor product is not *commutative*. That is, in general $X \otimes_h Y$ and $Y \otimes_h X$ are not isometric. We shall see some examples of this later.

One final property of the Haagerup tensor product: there are convenient norm expressions for $\|\cdot\|_{\mathrm{h}}$. Suppose that A and B are C^*-algebras, or spaces of the form

$B(K, H)$. If X and Y are subspaces of A and B respectively, and if $z \in X \otimes Y$, then by the definition in 1.5.4 and (1.4) we may write

$$\|z\|_{\mathrm{h}} \;=\; \inf \Big\| \sum_{k=1}^{p} a_k a_k^* \Big\|^{\frac{1}{2}} \Big\| \sum_{k=1}^{p} b_k^* b_k \Big\|^{\frac{1}{2}} \tag{1.42}$$

where the infimum is taken over all ways to write $z = \sum_{k=1}^{p} a_k \otimes b_k$ in $X \otimes Y$ (or equivalently, by the injectivity of this tensor product, in $A \otimes B$).

Proposition 1.5.6

(1) *If $z \in X \otimes_h Y$ with $\|z\|_{\mathrm{h}} < 1$ then we may write z as a norm convergent sum $\sum_{k=1}^{\infty} a_k \otimes b_k$ in $X \otimes_h Y$, with $\| \sum_{k=1}^{\infty} a_k a_k^* \| < 1$ and $\| \sum_{k=1}^{\infty} b_k^* b_k \| < 1$, and where the last two sums converge in norm. That is, $[a_1 \; a_2 \; \cdots] \in R(X)$ and $[b_1 \; b_2 \; \cdots]^t \in C(Y)$.*

(2) *If $x = [a_1 \;\; a_2 \;\; \cdots] \in R(X)$ and $y = [b_1 \;\; b_2 \;\; \cdots]^t \in C^w(Y)$, that is if $\sum_{k=1}^{\infty} a_k a_k^*$ converges in norm and if the partial sums of $\sum_{k=1}^{\infty} b_k^* b_k$ are uniformly bounded in norm, then $\sum_{k=1}^{\infty} a_k \otimes b_k$ converges in norm in $X \otimes_h Y$. Similarly if $x \in R^w(X)$ and $y \in C(Y)$.*

Proof (1) If z is as stated, choose $w_1 \in X \otimes Y$ with $\|z - w_1\|_{\mathrm{h}} < \frac{\epsilon}{2}$ and $\|w_1\|_{\mathrm{h}} < 1$. By (1.42) we may write $w_1 = \sum_{k=1}^{n_1} x_k \otimes y_k$ with $\sum_k x_k x_k^* \leq 1$ and $\sum_k y_k^* y_k \leq 1$. Repeating this argument, we may choose $w_2 \in X \otimes Y$ with $\|z - w_1 - w_2\|_{\mathrm{h}} < \frac{\epsilon}{2^2}$, and $\|w_2\|_{\mathrm{h}} < \frac{\epsilon}{2}$. By (1.42) we write $w_2 = \sum_{k=n_1+1}^{n_2} x_k \otimes y_k$ with $\sum_k x_k x_k^* \leq \frac{\epsilon}{2}$ and $\sum_k y_k^* y_k \leq \frac{\epsilon}{2}$. Continuing so, we obtain for every $m \in \mathbb{N}$ a finite rank tensor $w_m = \sum_{k=n_{m-1}+1}^{n_m} x_k \otimes y_k$ with $\|z - w_1 - \ldots - w_m\| < \frac{\epsilon}{2^m}$, $\sum_{k=n_{m-1}+1}^{n_m} x_k x_k^* \leq \frac{\epsilon}{2^{m-1}}$, and $\sum_{k=n_{m-1}+1}^{n_m} y_k^* y_k \leq \frac{\epsilon}{2^{m-1}}$. Now it is clear that the partial sums of $\sum_{k=1}^{\infty} x_k x_k^*$ and $\sum_{k=1}^{\infty} y_k^* y_k$ converge in norm to elements with norm close to 1, and that $z = \sum_{k=1}^{\infty} x_k \otimes y_k$ as desired.

(2) To see that the partial sums of $\sum_{k=1}^{\infty} a_k \otimes b_k$ are Cauchy, note that from (1.42) we have $\| \sum_{k=n}^{m} a_k \otimes b_k \|_{\mathrm{h}} \leq \| \sum_{k=n}^{m} a_k a_k^* \|^{\frac{1}{2}} \| \sum_{k=n}^{m} b_k^* b_k \|^{\frac{1}{2}}$. Now use the fact that the partial sums of $\sum_{k=1}^{\infty} a_k a_k^*$ are Cauchy. $\square$

The following important result is due to Christensen and Sinclair, and Paulsen and Smith. We will refer to it as the ‘CSPS theorem’. See the Notes section for references, and directions to a sample of the available proofs. Note that the second part of this result follows from the first part and 1.2.8.

Theorem 1.5.7 (CSPS theorem) *Suppose that X and Y are operator spaces, and that $u \colon X \times Y \to B(K, H)$ is a bilinear map.*

(1) *u is completely contractive (as a bilinear map) if and only if there is a Hilbert space L, and there are completely contractive linear maps $v \colon X \to B(L, H)$ and $w \colon Y \to B(K, L)$, with $u(x, y) = v(x)w(y)$ for all $x \in X$ and $y \in Y$.*

(2) *If X and Y are subspaces of unital C^*-algebras A and B respectively, and if the conditions in (1) hold, then there exist Hilbert spaces K_1 and K_2, unital*

$*$-representations $\pi\colon A \to B(K_1)$ and $\rho\colon B \to B(K_2)$, and contractions $T \in B(K, K_2), S \in B(K_2, K_1)$ and $R \in B(K_1, H)$, such that

$$u(x, y) \;=\; R\pi(x)S\rho(y)T, \qquad x \in X,\, y \in Y.$$

1.5.8 (Remarks on the CSPS theorem) An analoguous result to (1) holds for multilinear completely bounded maps of three or more variables, and can be proved using 1.5.7 and the discussion on associativity in 1.5.5. Thus if $X_1, \ldots, X_N$ are operator spaces and if $v_j\colon X_j \to B(H_j, H_{j-1})$ are completely contractive linear maps then the N-linear mapping taking $(x_1, \ldots, x_N) \in X_1 \times \cdots \times X_N$ to $v_1(x_1)\cdots v_N(x_N) \in B(H_N, H_0)$ is completely contractive. Conversely, any completely contractive N-linear map has this form.

Likewise, (2) has analoguous formulations for multilinear maps. For example, let $u\colon X \times Y \times Z \to B(K, H)$ be any completely contractive trilinear map, and assume that X and Z are subspaces of unital C^*-algebras A and B respectively. Then u has a factorization of the form $u(x, y, z) = R\pi(x)\Psi(y)\rho(z)T$, where $\pi\colon A \to B(K_1)$ and $\rho\colon B \to B(K_2)$ are unital $*$-representations, Ψ is a complete contraction from Y to $B(K_2, K_1)$, and R, T are contractions.

If $u\colon X \times Y \to B(K, H)$ is a completely contractive bilinear map, then the Hilbert space L in (1) may be replaced by the possibly smaller space which is densely spanned by $v(X)^*(H)$ and $w(Y)(K)$. In particular L may be chosen to be finite-dimensional if X, Y, K, H are all finite-dimensional.

As a final remark, we stress the case of completely bounded bilinear forms. If we have a completely contractive $u\colon X \times Y \to \mathbb{C}$, the CSPS theorem ensures that u may be written as $u(x, y) = \langle w(y), \overline{v(x)}\rangle_H$, where H is a Hilbert space and $v\colon X \to H^r$ and $w\colon Y \to H^c$ are completely contractive maps.

1.5.9 (Self-duality of $\otimes_h$) Let X and Y be operator spaces. Then

$$X^* \otimes_h Y^* \hookrightarrow (X \otimes_h Y)^* \quad \text{completely isometrically} \tag{1.43}$$

via the canonical map. Indeed, first assume that X and Y are finite-dimensional. We let J be the canonical map from $X^* \otimes_h Y^*$ into $(X \otimes_h Y)^*$, which is a surjection in our finite-dimensional case. By (1.8), it suffices to show that J_n is an isometric isomorphism from $M_n(X^* \otimes_h Y^*)$ to $CB(X \otimes_h Y, M_n)$, for any $n \geq 1$. We consider $z \in M_n(X^* \otimes_h Y^*)$ and we let $u = J_n(z)$. By 1.5.7 (1) and the third paragraph in 1.5.8, there exist an integer $p \geq 1$ and two linear maps $v\colon X \to M_{n,p}$ and $w\colon Y \to M_{p,n}$ such that $\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$ and $u(x \otimes y) = v(x)w(y)$ for any $x \in X$ and $y \in Y$. We let $\varphi = [\varphi_{ij}] \in M_{n,p}(X^*)$ and $\psi = [\psi_{ij}] \in M_{p,n}(Y^*)$ be the two matrices corresponding to v and w respectively. Then

$$u(x \otimes y) = v(x)w(y) = [\varphi_{ij}(x)][\psi_{ij}(y)] = \Big[\sum_k \varphi_{ik}(x)\psi_{kj}(y)\Big], \qquad x \in X, y \in Y.$$

Thus $z = \varphi \odot \psi$, and hence $\|z\|_{\mathrm{h}} \leq \|\varphi\|\|\psi\| = \|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$. The converse inequality $\|u\|_{\mathrm{cb}} \leq \|z\|_{\mathrm{h}}$ may be obtained by reversing the arguments.

In the general case, fix $[u_{ij}] \in M_n(X^* \otimes Y^*)$. Write each $u_{ij} \in X^* \otimes Y^*$ in the form $\sum_{k=1}^m \varphi_k \otimes \psi_k$, for functionals $\varphi_k \in X^*, \psi_k \in Y^*$. Let W (resp. Z) be the span of all these (finite number of) functionals in X^* (resp. Y^*), over all i and j too. We may view $[u_{ij}]$ in $M_n(W \otimes_h Z)$, and by the injectivity of $\otimes_h$ (see 1.5.5), its norm in the latter space equals its norm in $M_n(X^* \otimes_h Y^*)$. By the last paragraph, and by 1.4.4, its norm also equals the norm of the associated element in $M_n(((X/W_\perp) \otimes_h (Y/Z_\perp))^*)$. However, by the projectivity of $\otimes_h$ (see 1.5.5), and 1.4.3, we may view $((X/W_\perp) \otimes_h (Y/Z_\perp))^* \subset (X \otimes_h Y)^*$ completely isometrically. The composition of these identifications is easily seen to take $[u_{ij}]$ to the desired matrix in $M_n((X \otimes_h Y)^*)$, and so we are done.

1.5.10 (Some relations to Banach space tensor products) The Haagerup tensor product of minimal and maximal operator spaces corresponds to well-known Banach space tensor products (see A.3.5 for a review). Let E, F be Banach spaces. Then for any $m \geq 1$, we have isometric identities $R_m(\mathrm{Min}(E)) = \ell^2_m \check{\otimes} E$ and $C_m(\mathrm{Min}(F)) = \ell^2_m \check{\otimes} F$, as may be seen from (1.11). Then it is immediate from (1.40) and (A.5) that

$$\mathrm{Min}(E) \otimes_h \mathrm{Min}(F) \;=\; E \otimes_{\gamma_2} F \quad \text{isometrically.} \tag{1.44}$$

If G is an arbitrary Banach space and H is a Hilbert space then

$$CB(\mathrm{Min}(G), H^r) = \Pi_2(G, H) \quad \text{isometrically,} \tag{1.45}$$

where the latter denotes the Banach space of 2-summing maps from G into H (see A.3.4). To prove this consider $u\colon G \to H$ and assume that u is 2-summing. According to A.3.4, the mapping $I_{\ell^2_n} \otimes u\colon \ell^2_n \check{\otimes} G \to \ell^2_n(H)$ has norm less than or equal to $\pi_2(u)$ for any $n \geq 1$. Hence

$$\left\| I_{\ell^2_n} \otimes I_{\ell^2_n} \otimes u\colon \ell^2_n \check{\otimes} \ell^2_n \check{\otimes} G \longrightarrow \ell^2_n \check{\otimes} \ell^2_n(H) \right\| \leq \pi_2(u).$$

Using (1.2) and (A.1), we have isometric identities

$$M_n(H^r) \;\cong\; M_n(B(\bar{H}, \mathbb{C})) \;\cong\; B(\ell^2_n(\bar{H}), \ell^2_n) \;\cong\; \ell^2_n \check{\otimes} \ell^2_n(H).$$

On the other hand, $\ell^2_n \check{\otimes} \ell^2_n \cong M_n$ by (A.1) again, so $\ell^2_n \check{\otimes} \ell^2_n \check{\otimes} G = M_n(\mathrm{Min}(G))$ by (1.11). Thus the second last centered equation exactly means that the map u_n from $M_n(\mathrm{Min}(G))$ to $M_n(H^r)$ has norm $\leq \pi_2(u)$. Thus $\|u\|_{\mathrm{cb}} \leq \pi_2(u)$. The converse inequality is easily obtained by reversing the arguments.

Suppose that x in $R_m(\mathrm{Max}(E))$, viewed as an element of

$$R_m(\mathrm{Max}(E)^{**}) \cong CB(\mathrm{Max}(E)^*, R_m) = CB(\mathrm{Min}(E^*), R_m),$$

the last isomorphisms using 1.2.29 and (1.30). Applying (1.45) with $G = E^*$ we see that if $w\colon E^* \to \ell^2_m$ is the linear mapping corresponding to x, then $\|x\|$ is equal to $\pi_2(w)$. Therefore the argument before (1.44), and (A.6), show that

$$\mathrm{Max}(E) \otimes_h \mathrm{Min}(F) = E \otimes_{g_2} F \quad \text{isometrically.} \tag{1.46}$$

Likewise by (A.7) we have

$$\mathrm{Max}(E) \otimes_h \mathrm{Max}(F) = E \otimes_{\gamma_2^*} F \quad \text{isometrically.} \tag{1.47}$$

1.5.11 (Operator space projective tensor product) As with the Haagerup tensor product, it is convenient to first define an intimately related class of bilinear maps. Suppose that X, Y, and W are operator spaces and that $u\colon X \times Y \to W$ is a bilinear map. We say that u is *jointly completely bounded* if there exists a constant $K \geq 0$ such that

$$\|[u(x_{ij}, y_{kl})]_{(i,k),(j,l)}\| \leq K\|[x_{ij}]\|\|[y_{kl}]\|$$

for all m, n and $[x_{ij}] \in M_n(X)$, and $[y_{kl}] \in M_m(Y)$. The least such K is written as $\|u\|_{\mathrm{jcb}}$. We say that u is *jointly completely contractive* if $\|u\|_{\mathrm{jcb}} \leq 1$. Jointly completely bounded multilinear maps of three or more variables are defined similarly. We also observe that any completely contractive (in the sense of 1.5.4) bilinear map u is jointly completely contractive. This is immediate from the simple relation $[u(x_{ij}, y_{kl})] = u_{nm}([x_{ij}] \otimes I_m, I_n \otimes [y_{kl}])$.

Let X, Y be operator spaces, and let $n \in \mathbb{N}$. For $z \in M_n(X \otimes Y)$ define

$$\|z\|_{\frown} = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\|\}, \tag{1.48}$$

the infimum taken over $p, q \in \mathbb{N}$, and all ways to write $z = \alpha(x \otimes y)\beta$, where $\alpha \in M_{n,pq}, x \in M_p(X), y \in M_q(Y)$, and $\beta \in M_{pq,n}$. Here we wrote $x \otimes y$ for the 'tensor product of matrices', namely $x \otimes y = [x_{ij} \otimes y_{kl}]_{(i,k),(j,l)}$. We omit the easy proof (see [149] if necessary) that this defines an operator seminorm structure on $X \otimes Y$. For $z \in M_n(X \otimes Y)$, and for any jointly completely contractive bilinear map $u\colon X \times Y \to W$, it is easy to see from the definitions that

$$\|\tilde{u}_n(z)\| \leq \|z\|_{\frown}, \tag{1.49}$$

where $\tilde{u}\colon X \otimes Y \to W$ is the associated linear map. From the observation at the end of the last paragraph, it follows that this is also true if u is completely contractive. Taking $u = \otimes\colon X \times Y \to X \otimes_h Y$, we deduce that $\|\cdot\|_{\mathrm{h}} \leq \|\cdot\|_{\frown}$. Thus the quantities in (1.48) are norms. By Ruan's theorem the completion of $X \otimes Y$ with respect to these matrix norms is an operator space, which we call the *operator space projective tensor product*, and write as $X \widehat{\otimes} Y$.

By (1.48) and (1.49) we see that if $u\colon X \times Y \to W$ is a bilinear map with associated linear map $\tilde{u}\colon X \otimes Y \to W$, then u is jointly completely bounded if and only if $\tilde{u}$ extends to a completely bounded linear map on $X \widehat{\otimes} Y$. Moreover,

$$\|u\|_{\mathrm{jcb}} \;=\; \left\|\tilde{u}\colon X \widehat{\otimes} Y \longrightarrow W\right\|_{\mathrm{cb}}$$

in that case. The above property means that the operator space projective tensor product *linearizes jointly completely bounded bilinear maps.* As for the Haagerup tensor product this is a universal property, and characterizes $X \widehat{\otimes} Y$ uniquely up to complete isometry. From this it is easy to argue that

$$CB(X \widehat{\otimes} Y, Z) \cong CB(X, CB(Y, Z)) \cong CB(Y, CB(X, Z)) \tag{1.50}$$

isometrically, and indeed (by replacing Z by $M_n(Z)$) completely isometrically. In particular,

$$(X \widehat{\otimes} Y)^* \cong CB(X, Y^*) \cong CB(Y, X^*) \quad \text{completely isometrically.} \tag{1.51}$$

We now list a sequence of properties of the operator space projective tensor product. Again it is very easy to see, copying the idea of the analoguous proofs in 1.5.5, that $\widehat{\otimes}$ is *functorial*, and *projective*. It is also an easy exercise to see that $\widehat{\otimes}$ is *associative*, and (1.51) shows that $\widehat{\otimes}$ is *commutative*. We use these words in the sense that we have used them for the other two tensor products. From the universal properties of $\widehat{\otimes}$ and $\oplus_i^1$ (see above and 1.4.13), it is easy to check that

$$Y \widehat{\otimes} (\oplus_i^1 X_i) \;\cong\; \oplus_i^1 (Y \widehat{\otimes} X_i), \tag{1.52}$$

for operator spaces Y and $\{X_i\}$.

Proposition 1.5.12 *Let E, F be Banach spaces and let Y be an operator space.*

(1) $\mathrm{Max}(E) \widehat{\otimes} Y = E \hat{\otimes} Y$ *as Banach spaces.*

(2) $\mathrm{Max}(E \hat{\otimes} F) = \mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F)$ *as operator spaces.*

Proof The first item follows by computing the duals of these two tensor products, using (1.51), (1.12), and (A.4). Then (2) follows from (1) if we can show that $\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F)$ is a maximal operator space. To do this, observe that

$$B(\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F), W) = B(E \hat{\otimes} F, W) = B(E, B(F, W)),$$

for any operator space W, by (1) and (A.3). The latter space equals

$$CB(\mathrm{Max}(E), CB(\mathrm{Max}(F), W)) = CB(\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F), W)$$

by (1.12) and (1.50). Thus $\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F)$ is 'maximal'. □

Proposition 1.5.13 (Comparison of tensor norms) *If X and Y are operator spaces then the various tensor norms on $X \otimes Y$ are ordered as follows:*

$$\| \cdot \|_\vee \;\le\; \| \cdot \|_{\min} \;\le\; \| \cdot \|_{\mathrm{h}} \;\le\; \| \cdot \|_\frown \;\le\; \| \cdot \|_\wedge .$$

Indeed the 'identity' is a complete contraction $X \widehat{\otimes} Y \to X \otimes_h Y \to X \otimes_{\min} Y$.

Proof The first inequality follows from (1.34) and the definition of $\check{\otimes}$ (see A.3.1). The fact that $\| \cdot \|_\frown \le \| \cdot \|_\wedge$ follows from the universal property of $\hat{\otimes}$ (see A.3.3), since the bilinear map $\otimes \colon X \times Y \to X \widehat{\otimes} Y$ is jointly completely contractive, and hence contractive. We saw in 1.5.11 the complete contraction $X \widehat{\otimes} Y \to X \otimes_h Y$. For the remaining relation we may assume by the injectivity of these tensor norms that X and Y are unital C^*-algebras, and then $X \otimes_{\min} Y$

is a C^*-algebra as observed in 1.5.2. We may write $x \otimes y = (x \otimes 1)(1 \otimes y)$, which may be viewed as the product $\pi(x)\rho(y)$ of two (completely contractive) $*$-homomorphisms. Thus the bilinear map $\otimes\colon X \times Y \to X \otimes_{\min} Y$ is completely contractive, which by (1.41) proves the desired relation. $\square$

Proposition 1.5.14 *If X, Y are operator spaces, if H, K are Hilbert spaces, and if $m, n \in \mathbb{N}$, then we have the following complete isometries:*

(1) $H^r \otimes_h X = H^r \widehat{\otimes} X$, *and* $X \otimes_h H^c = X \widehat{\otimes} H^c$.
(2) $H^c \otimes_h X = H^c \otimes_{\min} X$, *and* $X \otimes_h H^r = X \otimes_{\min} H^r$.
(3) $C_n(X) \cong C_n \otimes_h X = C_n \otimes_{\min} X$, *and* $R_n(X) \cong X \otimes_h R_n = X \otimes_{\min} R_n$.
(4) $(\bar{H}^r \widehat{\otimes} X \widehat{\otimes} K^c)^* = (\bar{H}^r \otimes_h X \otimes_h K^c)^* \cong CB(X, B(K, H))$.
(5) $S^\infty(K, H) \cong H^c \otimes_{\min} \bar{K}^r$ *and* $S^\infty(K, H) \otimes_{\min} X \cong H^c \otimes_h X \otimes_h \bar{K}^r$.
(6) $M_{m,n}(X) \cong C_m \otimes_h X \otimes_h R_n$.
(7) $M_{m,n}(X \otimes_h Y) \cong C_m(X) \otimes_h R_n(Y)$.
(8) $H^c \widehat{\otimes} K^c = H^c \otimes_h K^c = H^c \otimes_{\min} K^c = (H \otimes^2 K)^c$, *and similarly for row Hilbert spaces.*
(9) $S^1(K, H) \cong \bar{K}^r \widehat{\otimes} H^c$.
(10) $CB(S^1(\ell^2_I, \ell^2_J), X) \cong \mathbb{M}_{I,J}(X)$, *if I, J are cardinals.*

Proof To prove that $X \otimes_h H^c = X \widehat{\otimes} H^c$ completely isometrically, it suffices by Proposition 1.5.13 to show that $I\colon X \otimes_h H^c \to X \widehat{\otimes} H^c$ is completely contractive. This will follow if we can show that any jointly completely contractive map $u\colon X \times H^c \to B(L)$ is completely contractive (in the sense of 1.5.4). Assume that $H = \ell^2_J$ for some cardinal J. Let $v\colon X \to R^w_J(B(L)) = B(L^{(J)}, L)$ be the linear map defined by $v(x) = (u(x, e_i))_{i \in J}$ for any $x \in X$. Then v corresponds to u via the identifications

$$CB(X \widehat{\otimes} C_J, B(L)) = CB(X, CB(C_J, B(L))) = CB(X, R^w_J(B(L)))$$

provided by (1.50) and Proposition 1.2.28. Thus $\|v\|_{\mathrm{cb}} = \|u\|_{\mathrm{jcb}}$. Then we define a map $w\colon C_J \to C_J(B(L)) \subset B(L, L^{(J)})$ by $w(\zeta) = (\zeta_j I_L)$ for $\zeta = (\zeta_j) \in \ell^2_J$. It is clear that $\|w\|_{\mathrm{cb}} = 1$ and moreover we have a factorization $u(x, \zeta) = v(x)w(\zeta)$ for any $x \in X, \zeta \in H = \ell^2_J$. By the easy part of the CSPS theorem 1.5.7, we deduce the desired inequality $\|u\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{jcb}}$. The first part of (1) is similar.

To prove (2), we may assume by injectivity of $\otimes_{\min}$ and $\otimes_h$ that H and X are finite-dimensional. In that case the result follows from (1) by duality, using (1.15), 1.5.9, (1.51), and (1.32). Item (3) follows from (2) and (1.37). The first equality in (4) is clear from (1), and the rest is clear from the string

$$(\bar{H}^r \widehat{\otimes} X \widehat{\otimes} K^c)^* \cong CB(X \widehat{\otimes} K^c, H^c) \cong CB(X, CB(K^c, H^c)),$$

which equals $CB(X, B(K, H))$. Here we have used (1.51), (1.50), (1.15), and (1.14). The first equality in (5) is clear from (1.32), (1.15), and (1.14). Then

$S^\infty(K,H) \otimes_{\min} X = H^c \otimes_{\min} X \otimes_{\min} \bar{K}^r$ by commutativity of $\otimes_{\min}$, and so the second part of (5) follows from (2). Item (6) is a special case of (5), and (7) follows from (6) by (3) and the associativity of the Haagerup tensor product. The middle equality in (8) follows from (2), and the first equality from (1). Writing $H^c = C_J$ and $\bar{K}^r = R_I$ for cardinals I, J, the last equality in (8) may be seen from (1.37) and (1.18). Clearly (9) follows from (4) with $X = \mathbb{C}$. Lastly, for (10), $CB(S^1(\ell^2_I, \ell^2_J), X)$ is completely isometric to

$$CB(R_I \widehat{\otimes} C_J, X) \cong CB(C_J, CB(R_I, X)) \cong R^w_J(C^w_I(X)) \cong \mathbb{M}_{I,J}(X),$$

using (9), (1.50), 1.2.28, and (1.20). □

1.6 DUALITY AND TENSOR PRODUCTS

This material is usually only used in the final section of later chapters.

1.6.1 (Mapping spaces as duals) If Y is a dual operator space then so is $CB(X,Y)$, for any operator space X. Indeed by (1.51) an explicit predual for $CB(X,Y)$ is $X \widehat{\otimes} Y_*$. From this, together with the density of the finite rank tensors in $X \widehat{\otimes} Y_*$, it follows that a bounded net $(u_t)_t$ in $CB(X,Y)$ converges in the w^*-topology to a $u \in CB(X,Y)$ if and only if $u_t(x)$ converges in the w^*-topology to $u(x)$ in Y for all $x \in X$. In particular, setting $Y = B(K,H)$ for Hilbert spaces H, K, and using the same principle again, we see that the above equivalent conditions are also equivalent to

$$\langle u_t(x)\zeta, \eta\rangle \;\to\; \langle u(x)\zeta, \eta\rangle \quad \text{for all} \quad x \in X, \zeta \in K, \eta \in H. \tag{1.53}$$

1.6.2 (Dual matrix spaces) If X is a dual operator space then so is $M_n(X)$. Indeed by (1.51) and (1.8) we have

$$(S^1_n \widehat{\otimes} X_*)^* \;\cong\; CB(X_*, M_n) \;\cong\; M_n(X).$$

More generally the same proof, but substituting 1.2.29 for (1.8), shows that for cardinals I, J, $\mathbb{M}_{I,J}(X)$ is a dual operator space with operator space predual $S^1(\ell^2_I, \ell^2_J) \widehat{\otimes} X_*$, and also $\mathbb{M}_{I,J}(X) \cong CB(X_*, \mathbb{M}_{I,J})$.

If I, J are sets, and if I_0 and J_0 are finite subsets of I and J respectively, write $\Delta = I_0 \times J_0$. The set Λ of such Δ is a directed set under the usual ordering. For such Δ, and for $x \in \mathbb{M}_{I,J}(X)$, we write x_Δ for the matrix x, but with entries x_{ij} switched to zero if $(i,j) \notin \Delta$. Then $(x_\Delta)_\Delta$ is a net indexed by $\Delta \in \Lambda$, which we call *the net of finite submatrices of* x.

Corollary 1.6.3 (Effros and Ruan) *Let X be a dual operator space, and let I, J be cardinals.*

(1) *If $(x_t)_t$ is a bounded net in $\mathbb{M}_{I,J}(X)$, then $x_t \to x \in \mathbb{M}_{I,J}(X)$ in the w^*-topology in $\mathbb{M}_{I,J}(X)$, if and only if each entry in x_t converges in the w^*-topology in X to the corresponding entry in x.*

(2) *If Y is a dual operator space, and if $u\colon X \to Y$ is a w^*-continuous completely bounded map, then the amplification $u_{I,J}\colon \mathbb{M}_{I,J}(X) \to \mathbb{M}_{I,J}(Y)$ is w^*-continuous. If, further, u is completely isometric, then $u_{I,J}$ is a w^*-homeomorphic complete isometry onto a w^*-closed subspace.*

(3) $\mathbb{M}^{\rm fin}_{I,J}(X)$ *is w^*-dense in $\mathbb{M}_{I,J}(X)$. Indeed if I, J are sets, and $x \in \mathbb{M}_{I,J}(X)$, then the net of finite submatrices of x converges to x in the w^*-topology.*

Proof Using the fact that $\mathbb{M}_{I,J}(X) = CB(X_*, \mathbb{M}_{I,J}) = CB(X_*, B(\ell^2_J, \ell^2_I))$ (see 1.6.2), (1) follows from (1.53), and the density of $\mathrm{Span}\{e_i\}$ in ℓ^2_I and ℓ^2_J. Items (2) and (3) follow immediately from (1) and A.2.5. $\square$

Note from 1.6.3 that for a w^*-closed subspace $X \subset B(K, H)$, one may *define* $\mathbb{M}_{I,J}(X)$ to be the w^*-closure of $\mathbb{M}^{\rm fin}_{I,J}(X)$ in $\mathbb{M}_{I,J}(B(K,H)) = B(K^{(J)}, H^{(I)})$.

1.6.4 (A characterization of dual operator spaces) An operator space X which is a dual Banach space, is a dual operator space if and only if $M_n(X)$ is a dual Banach space, and the n^2 canonical inclusion maps from X into $M_n(X)$ are w^*-continuous, for all $n \geq 2$. This follows immediately from the following criterion (which is the one which will be used in the sequel). If X is the dual Banach space of W, and if W is equipped with its natural matrix norms as a subspace of X^* via the natural inclusion, then X is the dual operator space of W if and only if the unit ball of $M_n(X)$ is $\sigma(X, W)$-closed for every positive integer n. That is, if and only if whenever $(x_t)_t$ is a net in $\mathrm{Ball}(M_n(X))$, $x \in M_n(X)$, and the i-j entry of x_t converges in the $\sigma(X, W)$ topology to the i-j entry of x for all $i, j = 1, \dots, n$, then $x \in \mathrm{Ball}(M_n(X))$.

The 'only if' in the last criterion follows from 1.6.2 and 1.6.3 (1). For the other direction, assume that the unit ball of $M_n(X)$ is $\sigma(X, W)$-closed for every $n \geq 1$. By 1.5.14 (6) and the selfduality of the Haagerup tensor product (see 1.5.9), we have $M_n(X)^* = R_n \otimes_h X^* \otimes_h C_n$. Since $\otimes_h$ is injective we deduce that $R_n \otimes_h W \otimes_h C_n \subset M_n(X)^*$ isometrically. Hence it is easy to argue that the image of $\mathrm{Ball}(M_n(X))$ inside $(R_n \otimes_h W \otimes_h C_n)^*$ has polar set in $R_n \otimes_h W \otimes_h C_n$ equal to the unit ball of $R_n \otimes_h W \otimes_h C_n$. Also, this image is w^*-closed by hypothesis. Therefore by the bipolar theorem, $\mathrm{Ball}(M_n(X))$ is equal to the unit ball of $(R_n \otimes_h W \otimes_h C_n)^*$. By 1.5.14 (4) and (1.8), the latter space is isometrically equal to $CB(W, M_n) = M_n(W^*)$. Hence $\mathrm{Ball}(M_n(X)) = \mathrm{Ball}(M_n(W^*))$. Since this holds for any $n \geq 1$, this shows that $X^* = W$ completely isometrically.

1.6.5 (Normal spatial tensor product) If X and Y are dual operator spaces, with operator space preduals X_* and Y_*, then $CB(Y_*, X)$ is the dual operator space of $X_* \widehat{\otimes} Y_*$ by 1.6.1. As in (1.35), we regard $X \otimes_{\rm min} Y \hookrightarrow CB(Y_*, X)$, and we define the *normal minimal tensor product* $X \overline{\otimes} Y$ to be the w^*-closure of $X \otimes Y$ (or of $X \otimes_{\rm min} Y$) in $CB(Y_*, X)$. Equivalently, if X and Y are w^*-closed subspaces of $B(H)$ and $B(K)$ respectively, then we may define $X \overline{\otimes} Y$ to be the w^*-closure in $B(H \otimes^2 K)$ of the copy of $X \otimes Y$. This is sometimes referred to as the *normal spatial tensor product*. If M and N are W^*-algebras, then $M \overline{\otimes} N$ as described above is the usual von Neumann algebra tensor product (e.g. see [407, IV; 5]) and in particular, $B(H) \overline{\otimes} B(K) = B(H \otimes^2 K)$. To see that these two definitions of $X \overline{\otimes} Y$ are the same (up to w^*-homeomorphic complete isometry), we use the following argument. Since X and Y are w^*-closed subspaces of $B(H)$ and $B(K)$ respectively, we know by 1.4.6 that X_* and Y_* are quotients of $S^1(H)$ and $S^1(K)$ respectively. By the 'projectivity' property of $\widehat{\otimes}$, we obtain a complete quotient map $Q\colon S^1(H) \widehat{\otimes} S^1(K) \to X_* \widehat{\otimes} Y_*$. Using the identification (1.51) we see that Q^* may be viewed as a w^*-continuous completely isometric embedding

$$CB(Y_*, X) \hookrightarrow CB(S^1(K), B(H)) \cong B(H \otimes^2 K),$$

the last relation from the first paragraph of 1.5.2. Via the canonical identification of $X \otimes Y$ with a subset of $CB(Y_*, X)$, we see that the w^*-closure of $X \otimes Y$ in $B(H \otimes^2 K)$ may be identified with the w^*-closure of $X \otimes Y$ in $CB(Y_*, X)$.

In general, $CB(Y_*, X)$ (or equivalently, $(X_* \widehat{\otimes} Y_*)^*$) is not equal to $X \overline{\otimes} Y$. However they are equal when X and Y are W^*-algebras, as was shown by Effros and Ruan (see the Notes section for a few more details). This is also the case when $X = B(H)$. Indeed, for any dual operator space Y and cardinals I, J,

$$\mathbb{M}_{I,J} \overline{\otimes} Y \cong \mathbb{M}_{I,J}(Y) \tag{1.54}$$

as dual operator spaces. This follows by the remark after 1.6.3, and an argument similar to the one used for (1.37). Also, $\mathbb{M}_{I,J}(Y) \cong CB(Y_*, \mathbb{M}_{I,J})$ by 1.2.29. Setting $H = \ell^2_I$ gives $B(H) \overline{\otimes} Y \cong CB(Y_*, B(H))$.

We leave it as an exercise that the normal spatial tensor product is 'associative', and 'functorial' for w^*-continuous completely bounded maps.

1.6.6 (Operator valued measurable functions) Let (Ω, μ) be a measure space (σ-finite for specificity), and let Y be a dual operator space. Since $L^\infty(\Omega)$ is a commutative W^*-algebra, $\|u\|_{\mathrm{cb}} = \|u\|$ for any $L^\infty(\Omega)$-valued bounded map (see 1.2.6). Using natural conditional expectations, there is a net v_λ of finite rank contractive maps on $L^\infty(\Omega)$ such that v_λ converges to I_{L^∞} point-w^*. If $u \in B(Y_*, L^\infty(\Omega))$ then $(v_\lambda u)_\lambda$ is a net of finite rank operators $Y_* \to L^\infty(\Omega)$ converging point-w^* to u. Since this net is bounded, this implies that $v_\lambda u \to u$ in the w^*-topology of $CB(Y_*, L^\infty(\Omega))$ by 1.6.1. Each $v_\lambda u$ is finite rank and so 'belongs' to $Y \otimes L^\infty(\Omega)$. This together with the first few lines of 1.6.5 shows that

$$B(Y_*, L^\infty(\Omega)) = CB(Y_*, L^\infty(\Omega)) \cong L^\infty(\Omega) \overline{\otimes} Y. \tag{1.55}$$

Assume that Y_* is separable. Then the latter operator space has an interesting description in terms of Y-valued functions defined on Ω. We say that a function $f\colon \Omega \to Y$ is w^*-measurable if $\langle f(\cdot), \varphi\rangle\colon \Omega \to \mathbb{C}$ is measurable for any $\varphi \in Y_*$. Using the separability assumption it is easy to see that for such an f, the scalar-valued function $\|f(\cdot)\|_Y$ is measurable. We set $\|f\|_\infty = \big\| \|f(\cdot)\|_Y \big\|_{L^\infty(\Omega)}$ and we say that f is essentially bounded if $\|f\|_\infty < \infty$. Then $\|f\|_\infty$ is a seminorm on the space of such functions and we let $L^\infty(\Omega; Y)$ be the normed space obtained after taking the quotient by $N_0 = \{f : \|f\|_\infty = 0\}$. As in the scalar-valued case, we identify any w^*-measurable essentially bounded function f with its class modulo N_0. Using again the separability of Y_*, we see that for any $f \in L^\infty(\Omega; Y)$,

$$\|f\|_\infty = \sup\bigl\{\|\langle f(\cdot\,), \varphi\rangle\|_{L^\infty} \,:\, \varphi \in \mathrm{Ball}(Y_*)\bigr\}.$$

Thus we may define a bounded map $J_f \colon Y_* \to L^\infty(\Omega)$, by $J_f(\varphi) = \langle f(\cdot\,), \varphi\rangle$, and we have $\|J_f\| = \|f\|_\infty$. It turns out that the resulting isometric embedding $L^\infty(\Omega; Y) \hookrightarrow B(Y_*, L^\infty(\Omega)) \cong (L^1(\Omega)\hat{\otimes}Y_*)^*$ is onto. This is proved e.g. in [380, Theorem 1.22.13] when Y is a von Neumann algebra and the proof given there extends to the general case. (See also Proposition IV.7.16 and its proof in [407].) The resulting identity $B(Y_*, L^\infty(\Omega)) \cong L^\infty(\Omega; Y)$ shows that the latter space is complete and by (1.55), we obtain an isometric identification

$$L^\infty(\Omega)\,\overline{\otimes}\,Y \;=\; L^\infty(\Omega; Y).$$

By the reasoning in 1.5.3 (3), this identity is a complete isometry if $L^\infty(\Omega; Y)$ has matrix norms obtained by equating $M_n(L^\infty(\Omega; Y))$ and $L^\infty(\Omega; M_n(Y))$. Thus $L^\infty(\Omega; Y)$ is a dual operator space. If $j_Y \colon L^\infty(\Omega)\,\overline{\otimes}\,Y \to L^\infty(\Omega; Y)$ is the w^*-continuous mapping providing this identification, and if we consider finite families $(f_k)_k$ in $L^\infty(\Omega)$ and $(y_k)_k$ in Y, then j_Y maps $\sum_k f_k \otimes y_k$ to $\sum_k f_k(\cdot)y_k$.

1.6.7 (W^*-continuous extensions: the bilinear case) Let X, Y be operator spaces, let W be a dual operator space, and let $u \colon X \times Y \to W$ be a completely bounded bilinear map. Then u admits a (necessarily unique) separately w^*-continuous extension $\tilde{u} \colon X^{**} \times Y^{**} \to W$, this extension is completely bounded, and $\|\tilde{u}\|_{\mathrm{cb}} = \|u\|_{\mathrm{cb}}$. To prove this, we may assume by Lemma 1.4.7 that W is a w^*-closed subspace of some $B(H)$. By the CSPS theorem (1.5.7), we find a Hilbert space L and two completely bounded maps $v \colon X \to B(L, H)$ and $w \colon Y \to B(H, L)$ such that $u(x, y) = v(x)w(y)$ for all $x \in X$, $y \in Y$. According to 1.4.8, v and w admit w^*-continuous extensions $\tilde{v} \colon X^{**} \to B(L, H)$ and $\tilde{w} \colon Y^{**} \to B(H, L)$ with $\|\tilde{v}\|_{\mathrm{cb}} = \|v\|_{\mathrm{cb}}$ and $\|\tilde{w}\|_{\mathrm{cb}} = \|w\|_{\mathrm{cb}}$. Define a bilinear map $\tilde{u} \colon X^{**} \times Y^{**} \to B(H)$ by letting $\tilde{u}(\eta, \nu) = \tilde{v}(\eta)\tilde{w}(\nu)$ for any $\eta \in X^{**}$, $\nu \in Y^{**}$. Then the easy part of the CSPS theorem ensures that $\tilde{u}$ is completely bounded with $\|\tilde{u}\|_{\mathrm{cb}} \leq \|\tilde{v}\|_{\mathrm{cb}}\|\tilde{w}\|_{\mathrm{cb}} = \|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$. Since u is valued in W, and W is w^*-closed, its extension $\tilde{u}$ is valued in W as well.

1.6.8 (Normal Haagerup tensor product) If X and Y are two dual operator spaces, we let $(X \otimes_h Y)^*_\sigma$ denote the subspace of $(X \otimes_h Y)^*$ corresponding to the completely bounded bilinear forms $X \times Y \to \mathbb{C}$ which are separately w^*-continuous. Then we define the *normal Haagerup tensor product* $X \otimes_{\sigma h} Y$ to be the operator space dual of $(X \otimes_h Y)^*_\sigma$.

From 1.6.7 one may deduce the following two consequences. First, if X, Y are two (not necessarily dual) operator spaces then $(X \otimes_h Y)^* \cong (X^{**} \otimes_h Y^{**})^*_\sigma$. To see this, consider the map $u \mapsto \tilde{u}$ in 1.6.7, in the case that $W = M_n$. Note that a map u into M_n is separately w^*-continuous if and only if each 'entry' u_{ij} is separately w^*-continuous. It follows that

$$(X \otimes_h Y)^{**} = X^{**} \otimes_{\sigma h} Y^{**} \quad \text{as dual operator spaces.} \tag{1.56}$$

Second, if X, Y are two dual operator spaces, then the natural action of $X \otimes Y$ on $(X \otimes_h Y)^*_\sigma$ yields a completely isometric embedding

$$X \otimes_h Y \ \subset\ X \otimes_{\sigma h} Y. \tag{1.57}$$

To see (1.57), let $j\colon X \otimes Y \to X \otimes_{\sigma h} Y$ be the canonical mapping, and fix z in $X \otimes Y$. It is clear from the definitions that $\|j(z)\| \leq \|z\|_{\mathrm{h}}$. The converse inequality relies on the selfduality of the Haagerup tensor product. We let $\epsilon > 0$. By (1.43), we have $X \otimes_h Y \subset (X_* \otimes_h Y_*)^*$ isometrically. Hence there exists a map $\Phi \in X_* \otimes Y_*$ such that $\|\Phi\|_{\mathrm{h}} = 1$ and $\|z\|_{\mathrm{h}} \leq |\langle z, \Phi\rangle| + \epsilon$. Writing Φ as $\sum_k \varphi_k \otimes \psi_k$ for some $\varphi_k \in X_*$ and $\psi_k \in Y_*$, then the bilinear form $u\colon X \times Y \to \mathbb{C}$ defined by $u(x, y) = \sum_k \varphi_k(x)\psi_k(y)$ is clearly separately w^*-continuous and we have $\langle z, \Phi\rangle = \langle j(z), u\rangle$. Hence $|\langle z, \Phi\rangle| \leq \|j(z)\|\|u\|_{\mathrm{cb}}$. By (1.43) again, and the injectivity of $\otimes_h$, we have $X_* \otimes_h Y_* \subset X^* \otimes_h Y^* \subset (X \otimes_h Y)^*$, so that $\|u\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{h}} = 1$. Since ϵ was arbitrary, this shows that $\|z\|_{\mathrm{h}} \leq \|j(z)\|$. Hence j is an isometry, and a similar argument shows that it is a complete isometry.

Indeed, $X \otimes Y$ is w^*-dense in $X \otimes_{\sigma h} Y$. Let $q\colon (X \otimes_h Y)^{**} \to X \otimes_{\sigma h} Y$ be the adjoint mapping of the embedding $(X \otimes_h Y)^*_\sigma \subset (X \otimes_h Y)^*$, and let $i\colon X \otimes_h Y \to (X \otimes_h Y)^{**}$ be the canonical embedding into the second dual. Then the embedding in (1.57) is qi. That its range is w^*-dense follows from Goldstine's lemma A.2.1, and the fact that q is a quotient map, and is w^*-continuous.

1.6.9 (Weak* Haagerup tensor product) This is another 'dual version' of the Haagerup tensor product. Since will not use this tensor product very much, we content ourselves with an abridged development of it. See [70, 150] for much more information, for example, there are several other ways to describe this tensor product. If X and Y are operator spaces then we define

$$X^* \otimes_{w^* h} Y^* \ =\ (X \otimes_h Y)^*. \tag{1.58}$$

We now discuss why this is a 'tensor product'. By the last paragraph of 1.5.8, and writing the space H there as ℓ^2_I, we have that $w \in (X \otimes_h Y)^*$ if and only if w may be written in the form $w(x \otimes y) = \langle \varphi(x), \psi(y)\rangle$, for a cardinal I, $\varphi \in CB(X, R_I)$ and $\psi \in CB(Y, C_I)$. Now $CB(Y, C_I) \cong C^w_I(Y^*)$ and $CB(X, R_I) = R^w_I(X^*)$, using 1.2.29. Writing $\varphi = [\varphi_i]$ and $\psi = [\psi_i]$, we have shown the following assertion. Namely, $w \in (X \otimes_h Y)^*$ if and only if there exist $[\varphi_i] \in R^w_I(X^*)$ and $[\psi_i] \in C^w_I(Y^*)$ such that w may be written as $w(x \otimes y) = \sum_i \varphi_i(x)\psi_i(y)$ for $x \in X, y \in Y$. The last sum converges absolutely in $\mathbb{C}$, as may be seen by the Cauchy–Schwarz inequality. We may therefore think of elements $w \in X^* \otimes_{w^* h} Y^*$

as tensors, and write $w = \sum_i \varphi_i \otimes \psi_i$, or $w = \varphi \odot \psi$. We call this a *weak representation* for w, and equate two weak representations if they agree 'as functionals on $X \otimes_h Y$'. By the above,

$$\|w\| = \min\{\|[\varphi_i]\|_{R_I^w(X^*)} \|[\psi_i]\|_{C_I^w(Y^*)}\},$$

where the minimum is over all weak representations for w.

Note that $X \otimes_h Y \subset X \otimes_{w^*h} Y$ completely isometrically. This is just a restatement of (1.43). Thus if X is finite-dimensional then $X \otimes_{w^*h} Y = X \otimes_h Y$. It follows readily from the 'associativity' of the Haagerup tensor product (see 1.5.4), and (1.58), that the weak* Haagerup tensor product is associative too. It is 'functorial' too in an appropriate sense. That is, w^*-continuous completely bounded maps will tensor. In fact more is true: Given dual operator spaces X_k and Y_k, for $k = 1, 2$, and completely bounded maps $u_k \colon X_k \to Y_k$, there is a canonical completely bounded map $u_1 \otimes u_2 \colon X_1 \otimes_{w^*h} X_2 \to Y_1 \otimes_{w^*h} Y_2$ with $(u_1 \otimes u_2)(\sum_i x_i \otimes w_i) = \sum_i u_1(x_i) \otimes u_2(w_i)$, for any weak representation $\sum_i x_i \otimes w_i$. Moreover, $\|u_1 \otimes u_2\|_{\mathrm{cb}} \leq \|u_1\|_{\mathrm{cb}} \|u_2\|_{\mathrm{cb}}$. We omit the proof of this perhaps surprising assertion, which may be found in the original paper [70] (or see [150] for a different proof).

Many of the identities in 1.5.14 have versions appropriate to $\otimes_{w^*h}$. For example, for a dual operator space X and a cardinal I we have

$$C_I \otimes_{w^*h} X = C_I \bar{\otimes} X \cong C_I^w(X) \tag{1.59}$$

and a similar relation holds for $R_I^w(X)$. Indeed, the '$\cong$' here follows from (1.54). The predual of $C_I^w(X)$ is $R_I \widehat{\otimes} X_*$, by 1.2.28 and (1.51). However by 1.5.14 (1), the latter space equals $R_I \otimes_h X_*$, which is the predual of $C_I \otimes_{w^*h} X$.

Another useful relation is

$$R_I \otimes_{w^*h} X = R_I \otimes_h X = R_I \widehat{\otimes} X \cong C_I(X_*)^* \tag{1.60}$$

(and there is a matching formula for $R_I(X_*)^*$). To see (1.60), note that if w is in $R_I \otimes_{w^*h} X$, then w has a weak representation $\sum_i r_i \otimes x_i$, with $[r_i] \in R_J^w(R_I)$ and $[x_i] \in C_J^w(X)$, for a cardinal J. However $R_J^w(R_I) = R_J(R_I)$, so that by the proof of 1.5.6 (2) we have that $w \in R_I \otimes_h X$. This gives the first equality in (1.60). The second equality is 1.5.14 (1), whereas the third follows from (1.58), since $C_I(X_*) = C_I \otimes_h X_*$ by 1.5.14 (2).

By analogy with 1.5.14 (7), for cardinals I, J we have

$$M_{I,J}(X \otimes_{w^*h} Y) \cong C_I^w(X) \otimes_{w^*h} R_J^w(Y). \tag{1.61}$$

To see this, notice that the first space here equals

$$C_I \otimes_{w^*h} (X \otimes_{w^*h} Y) \otimes_{w^*h} R_J \cong (C_I \otimes_{w^*h} X) \otimes_{w^*h} (Y \otimes_{w^*h} R_J),$$

using (1.20), (1.59) and its matching 'row version', and the associativity of $\otimes_{w^*h}$. By (1.59) again, this equals the right-hand side of (1.61).

As an application, we show that for any operator space X we have

$$\mathbb{K}_{I,J}(X)^{**} \cong \mathbb{M}_{I,J}(X^{**}) \quad \text{as dual operator spaces.} \tag{1.62}$$

By 1.5.14 (5), (1.37), (1.58), and (1.60) and its matching 'column' formulation,

$$\mathbb{K}_I(X)^* \cong (C_I \otimes_h X \otimes_h R_I)^* \cong (C_I \otimes_h X)^* \otimes_h C_I \cong R_I \otimes_h X^* \otimes_h C_I.$$

By 1.5.14 (4) and 1.2.29 we have $(R_I \otimes_h X^* \otimes_h C_I)^* \cong CB(X^*, \mathbb{M}_I) \cong \mathbb{M}_I(X^{**})$. This proves (1.62). Another proof may be given using C^*-algebraic principles.

1.6.10 (A principle for separately w^*-continuous maps) Suppose, for example, that we have a completely bounded trilinear map $\Phi\colon X \times Y \times Z \to B(K, H)$, where X, Y, Z are dual operator spaces. Suppose that Φ is separately w^*-continuous. From the CSPS theorem 1.5.7 we have

$$\Phi(x, y, z) \;=\; u(x)v(y)w(z) \tag{1.63}$$

for completely contractive maps $u\colon X \to B(K_1, H)$, $v\colon Y \to B(K_2, K_1)$, and $w\colon Z \to B(K, K_2)$. Here of course H, K, K_1, K_2 are Hilbert spaces. Then we claim that K_1, K_2, and the maps u, v, w may be slightly adjusted, to each be w^*-continuous and still satisfy (1.63).

To see this, we first state an obvious principle, which we leave as an exercise. Namely, if $u\colon X \to B(K, H)$ is a completely contractive w^*-continuous map, if L is a subspace of H with associated projection P_L, and if G is a subspace of K, then the map $P_L u(\cdot)_{|G}$ is w^*-continuous and completely contractive too.

Because of this principle, the following procedure applied to the maps one at a time, from right to left, will make them w^*-continuous, without destroying w^*-continuity of any of the previously adjusted maps. We begin on the right, with w. We note that for $\zeta \in K, \eta \in H$,

$$\langle \Phi(x, y, z)\zeta, \eta\rangle = \langle w(z)\zeta, v(y)^* u(x)^* \eta\rangle = \langle P_L w(z)\zeta, v(y)^* u(x)^* \eta\rangle, \tag{1.64}$$

where $L = [v(Y)^* u(X)^* H]$, and P_L is the projection onto L. Replacing w by $P_L w$, and v by its restriction to L, we may assume that $K_2 = L$. If $(z_t)_t$ is a bounded net in Z converging in the w^*-topology to z, and if $x \in X$ and $y \in Y$ are fixed, then $\Phi(x, y, z_t) \to \Phi(x, y, z)$ in the w^*-topology. Thus from (1.64),

$$\langle w(z_t)\zeta, v(y)^* u(x)^* \eta\rangle \to \langle w(z)\zeta, v(y)^* u(x)^* \eta\rangle.$$

From this, and norm density considerations in L, it is clear that $w(z_t) \to w(z)$ in the WOT. Since this is a bounded net, it converges in the w^*-topology (see A.1.4). Thus by A.2.5, w is w^*-continuous. Finally, we replace K_2 by its subspace $[w(Z)H]$, and replace v by its restriction to this subspace. This ends the procedure. We now begin the same procedure again, to make the next map v w^*-continuous. Thus we replace K_1 by its subspace $[u(X)^* H]$, and use an argument similar to the above. Once v is w^*-continuous, we replace K_1 again by $[v(Y)w(Z)H]$, and then continue, to make u w^*-continuous.

1.7 NOTES AND HISTORICAL REMARKS

One might say that operator space theory grew out of Stinespring's theorem [398], via Arveson's landmark papers [21, 22] and the fundamental work in the seventies (continuing into the early 1980s) of Choi and Effros, Haagerup, Kirchberg, Wittstock, and others. The eighties saw much activity in the area of completely bounded maps; the vision of Effros (e.g. see [133]) and Paulsen's text [307] were particularly influential in this development. See also [94], and the much overlooked work of Hamana (referenced in our bibliography). However operator spaces did not become a field in its own right until Ruan's theorem 1.2.13. This result was proved by Ruan in his thesis [369] (see also [370]) under the direction of Effros, and was inspired by [90,318]. Ruan used this theorem to define operator space quotients, operator seminorm structures, and so on. See also the inspiring papers [142, 143], for example. A couple of years after that, operator space duality, and the basic operator space tensor theory, were developed independently by Blecher and Paulsen, and Effros and Ruan (e.g. see [42, 66, 145, 146]). (The Haagerup tensor product of operator spaces had been developed earlier (e.g. in the important paper [318]).) In the early 1990s Pisier lent his considerable strength and expertise to the subject, as is fabulously testified to in [332, 337]. After this the field grew quite considerably, with many brilliant mathematicians making remarkable contributions. For a more complete account of basic operator space theory, and for more references to the literature to complement the list in these Notes and historical remarks sections, the reader should also consult the texts [149, 314, 337, 385]. See also [433], which is currently being rewritten in an expanded printed form. There are other good surveys of part of the subject, such as [338, 366].

1.2: Parts of 1.2.4 are folklore, no doubt known from the beginning. A generalization is proved in the later result 8.3.2. Observation 1.2.6 is due to Loebl. A related useful fact is that for a bounded linear map into M_n, we have $\|T\|_{\mathrm{cb}} = \|T_n\|$ (see [389]). The canonical example of a map which is not completely bounded is the 'transpose map' on $\mathbb{K}$. Injectivity of operator systems was studied deeply by Choi and Effros (e.g. see [90]), partly inspired by Arveson's extension theorem (see 1.3.5 and [21]). Later, Theorems 1.2.8 and 1.2.10 were proved by Wittstock [432, 431], Haagerup [177], and Paulsen [305]. See those papers, and [149, 314, 337, 385], for additional historical details. Paulsen's proofs of these two results is now the standard route. See also [325, 335] for a different approach extending Theorem 1.2.8 to a Banach space setting. The first isomorphism in (1.5) has been dubbed the 'canonical shuffle' by Paulsen. It is not true that a one-to-one and surjective completely bounded linear map $T\colon X \to Y$ between complete operator spaces has a completely bounded inverse. Indeed, there is no 'open mapping theorem' in this sense. Of course if the countably infinite amplification of T maps $\mathbb{K}(X)$ onto $\mathbb{K}(Y)$, then there is no obstacle.

That $CB(X,Y)$ is an operator space was observed first in [143]. The operator space dual was called the *standard dual* by Blecher and Paulsen, and the associated duality theory was developed independently by those authors and Effros and Ruan. Minimal and maximal operator spaces were first considered in [142] and [66] respectively. Formally, Hilbert row and column spaces may first appear in [432]. They played a role in [135,66] and were further developed in [147]. Some of the results in the latter paper had also been noticed by Blecher, and [43] gives alternative routes to these and to other results from [147]. There are other interesting operator space structures on a Hilbert space, most notably the (selfdual) operator Hilbert space OH of Pisier [328, 337] which will be mentioned again in Chapter 5 (see 5.3.4). Nearly all of the definitions/results on infinite matrix spaces here may be found in a series of papers by Hamana, and Effros and Ruan (e.g. see [186,143,146]). Interpolation and ultraproducts of operator spaces were first considered by Pisier [328,331,327]. See also [436] for related work.

We sketch a simple proof of (1.14). We may write $H^c = C_J, K^c = C_I$, for cardinals I, J. Setting $X = C_I$ in 1.2.28 yields $CB(C_J, C_I) \cong R_J^w(C_I) \cong \mathbb{M}_{I,J}$ completely isometrically, where the last equality follows from any of the centered equations in 1.2.26.

1.3: Many of the results in this section are from Arveson's original papers [21, 22]. These papers established, for example, the role of completely positive maps in the unitary equivalence theory of irreducible sets of compact operators, a generalization of the Shilov boundary to arbitrary operator algebras, and generalizations of the Sz. Nagy–Foias dilation theorem to representations of arbitrary function algebras. Arveson's result 1.3.6 is the decisive relation between unital operator spaces and operator systems. Paulsen's lemma 1.3.15 (originally from [305]) yields the decisive relation between general operator spaces and operator systems. Note that we did not use the fact that the map was unital in 1.3.9, indeed many of the results in this section are true without this assumption (e.g. see [36]). There are also direct proofs of 1.3.9 not using Stinespring's theorem. The result 1.3.10 is an easy variant of the Banach–Stone–Kadison theorem [212]. Item 1.3.11 seems to appear first in [301], but is related to Choi's multiplicative domain [87]. Other sources for results in this section are [88,90,211,213,307,399].

1.4: The facts in this section are due to Blecher (see [42], which was written close to the date of [66,145], although it appeared much later), with the following main exceptions. The fact that $X \subset X^{**}$ completely isometrically was independently noticed in [66, 145]. Effros and Ruan had noticed 1.4.7 via a different route [146].

1.5: The minimal tensor product may have been first considered for operator spaces in [307]. An account of this tensor product from the perspective of C^*-algebra theory can be found in [407], for example. Relations such as the complete isometry $X \otimes_{\min} Y \hookrightarrow CB(Y^*, X)$ were first noticed in [66]. The approach presented here to the minimal tensor product is taken from the latter paper, with some simplifications influenced by the exposition in Chapter 8 of [149]. The functoriality property of the minimal (or spatial) tensor product is one of the most important properties of completely bounded maps: they may be tensored. This may have been first observed in [117], which dates to 1982 or earlier. This property is not shared by general bounded maps with respect to the natural tensor products found in C^*-algebra theory.

Haagerup defined the tensor product that bears his name in [176], at least for C^*-algebras. It was studied in the 1980s in papers of Effros, some with coauthors (see [149] for detailed references). In [318] Paulsen and Smith showed that the Haagerup tensor product in the generality presented here, is an operator space (this is usually seen now using Ruan's theorem, as was first noted in [369]), and proved many of the other facts presented here. The class of bilinear or multilinear maps described here as 'completely bounded' are given a different name in [149]; instead Effros and Ruan use this term for the class which we call 'jointly completely bounded'. The injectivity of the Haagerup tensor product is also from [318]. Other proofs were found later by Effros, Paulsen, and others (e.g. see [385, 149, 314, 337] or 2.10 in [43], for a selection). The projectivity of $\otimes_h$ and $\widehat{\otimes}$ was observed in [147]. Part (2) of 1.5.6 is also true for uncountable sums with the same proof. The original references for Theorem 1.5.7 are [93] (for C^*-algebras) and [318] (for general operator spaces). Other proofs may be found in [149,307,337]. The selfduality relation (1.43) was conjectured by Blecher and proved in full in [147]. Different proofs appear in [43,70]. The latter paper sketches a route based on Smith's important notion of *strong independence* [391], through many of the basic results in this section. Observation (1.44) was perhaps first in [40]. The relation (1.45) is from [147], see also [325] and [310] for related results and developments. For further information on g_2, we refer the reader to [118] (e.g. see 12.7 there). Relations (1.44), (1.46), and (1.47) are just three examples of relationships between the Haagerup tensor product and Pisier's 'gamma-norms' introduced in [326]. See [328] and [243] for related developments.

The operator space projective tensor product and its basic properties were developed in [66,145]. Formula (1.48) is from the latter paper. There is a similar formula valid for all $z \in M_n(X \widehat{\otimes} Y)$, writing $z = \alpha(x \otimes y)\beta$, where the latter matrices are now countably infinite (see [149]). In [66] it is shown without using formula (1.48), that there is an operator space $X \widehat{\otimes} Y$ having all the other properties mentioned in 1.5.11 and the paragraphs after it. Various forms of the relations in 1.5.14 may be found in [135,66,147,139,43], but [147] is the primary reference. Note that 1.2.10 follows from 1.5.14 (4), and the injectivity of $\otimes_h$.

1.6: Nearly all the facts about infinite matrices in 1.6.2 and 1.6.3 are explicitly in [186, 143, 146]. Neufang has generalized 1.6.3 in [292]. The result 1.6.4 is due to Le Merdy [244], who also showed in that paper that if X is an operator space which is a dual Banach space with predual Y, then there may not exist an operator space structure on Y such that $Y^* \approx X$ completely isomorphically (see also 2.7.15). The equivalence between the two descriptions of $\overline{\otimes}$ in 1.6.5 is related to the fact that $(X_* \widehat{\otimes} Y_*)^* = CB(Y_*, X)$ equals the *normal Fubini product* of X and Y, which was proved independently by Blecher and Ruan (see [43, Theorem 2.5], [372]), inspired by the von Neumann algebra case [146]. The Fubini product for operator spaces was first studied in [186] though. Since the normal Fubini product is defined in terms of *slice maps*, such maps are key to understanding when $X \overline{\otimes} Y = CB(Y_*, X)$ holds for dual operator spaces X and Y. For example, the fact that this holds for two von Neumann algebras, which was proved in [146], follows from the slice map theorem of Tomiyama [410]. This fact is of great importance in many recent papers on Fourier analysis (e.g. see [378]), since, for example, one may deduce from it that $A(G_1 \times G_2) \cong A(G_1) \widehat{\otimes} A(G_2)$ for locally compact groups G_1, G_2 (see [146]). A dual operator space X is said to have the dual slice mapping property if $X \overline{\otimes} Y = CB(Y_*, X)$ holds for all dual operator spaces Y. This is equivalent to X_* possessing the 'operator space approximation property' (see [146,139,233]). The argument at the beginning of 1.6.6 is only part of this result in the case when X is a commutative W^*-algebra. It extends to all semidiscrete (=injective) von Neumann algebras. The description of $L^\infty(\Omega; Y)$ as a space of Y-valued w^*-measurable functions is essentially taken from [380, Section 1.22]. The definitions and results in 1.6.8 are due to Effros and Ruan (see [138, 150], and references therein). The weak* Haagerup tensor product was developed by Blecher and Smith in [70] for dual operator spaces. Later this tensor product, and its basic properties, were generalized to all operator spaces (e.g. see [150], and to a lesser extent [5]), under the name *extended Haagerup tensor product*. Spronk has shown in [397] that the original arguments of [70] immediately yield many of these generalizations from [150]. Result 1.6.10 is from [150].