1

GENESIS OF QUANTUM ANOMALIES

1.1 Introduction

The central dogma in modern physics is the principle of quantum theory. All dynamics, either classical or relativistic dynamics, or all the forces such as electromagnetic and nuclear forces need to be formulated to conform to the principle of quantum theory when applied to microscopic systems. The general theory of relativity is not exceptional, and quantum gravity needs to be formulated in microscopic domains. Even classical dynamics is realized as a vanishing limit of the Planck constant $\hbar$ in quantum theory.

The basic equation of quantum theory is given by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi \tag{1.1}$$

and the wave function ψ has a meaning as the probability amplitude. The wave function is an element of infinite-dimensional Hilbert space, as is expected from the fact that the wave function of a hydrogen atom has an infinite number of components. To emphasize this aspect, the wave function is written as the state vector $|\psi\rangle$. From the definition of probability, the square of the state vector needs to be non-negative, namely, we postulate that the norm of the wave function be positive definite

$$\langle\psi|\psi\rangle > 0. \tag{1.2}$$

The time development of the Schrödinger equation is described by a hermitian Hamiltonian, and thus the time development induces a unitary transformation of the wave function

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}Ht\right)|\psi(0)\rangle. \tag{1.3}$$

By definition, the unitary transformation does not change the magnitude (total probability) of the wave function. What changes is, for example, that the first component is transformed to the second component of an infinite-dimensional state vector.

To apply quantum theory to general phenomena including photons, it is necessary to quantize the field variable itself and thus formulate field theory. Intuitively, one distributes oscillators at each point of four-dimensional space-time in field theory, and quantizes all these oscillators. The excitation or de-excitation of those oscillators are interpreted as the creation or annihilation of associated particles. There are an infinite number of points and thus one needs to handle an infinite number of oscillators, or if one formulates the problem suitably, one deals with a system where an infinite number of harmonic oscillators interact with each other. For this reason, the field theory is called a quantum theory of an infinite number of degrees of freedom. We have an infinite-dimensional Hilbert space even for a single harmonic oscillator, and now we have an infinite number of oscillators.

A representative field theory is quantum electrodynamics which deals with the electron and the photon and their interactions. As is expected, researchers encountered various subtle problems when they first attempted to describe the creation and annihilation of particles. Among them, the divergence problem in field theory is the best known. A careful examination of the divergence problem associated with creation and annihilation led to the discovery of the phenomenon called the *quantum anomaly* or simply *anomaly*, which is the main subject of this book. The essential difference between the quantum anomaly and divergences is that the quantum anomaly does not diverge, though both of them are related to an infinite number of degrees of freedom. It is proper to understand the quantum anomaly as a symmetry breaking by the quantization procedure. However, the quantum anomaly usually appears in a form closely related to divergences in interaction picture perturbation theory. For this reason, it is often said that the quantum anomaly is associated with divergences and their regularization. This way of characterizing the anomaly correctly reflects the historical origin of the quantum anomaly (and its certain aspects), but one should rigorously distinguish the quantum anomaly from divergences.

1.2 Is the photon massless?

We here briefly describe the history of the study of the quantum anomaly. Quantum electrodynamics (QED) and its renormalization formulated in the late 1940s indicated that the applicability domain of the principle of quantum theory is quite broad and possibly covers all the energy ranges we can think of. This quantum electrodynamics, as is well known, was formulated by S. Tomonaga, J. Schwinger, R. Feynman, F. Dyson and others. In particular, the relativistic formulation of quantum electrodynamics by Tomonaga and his associates in isolated Japan after the Second World War was a surprise to physicists in the United States, in addition to the meson theory of H. Yukawa. J. Oppenheimer, who was director of the Institute for Advanced Study at Princeton at that time, asked Tomonaga to send a summary of the research of his group to him, which was later published as a Letter to the Editor in the Physical Review. In this letter, Tomonaga explained that covariant quantum electrodynamics and its renormalization prescription can deal successfully with all the problems associated with the electron mass, the electric charge, and the field variables of the electron and the photon, but he encountered a difficulty with the photon mass. He stated, “But for this subtraction we cannot find a reasoning so natural and plausible as that used in the case of mass-type and charge-type infinities, where the subtraction was considered as an amalgamation. This is because it would necessarily result in a drastic change of the Maxwell equation for the radiation”.

If this difficulty should persist, it suggested that we may not be able to handle the experimentally established vanishing photon mass and the gauge principle, which ensures the vanishing photon mass, in interaction picture perturbation theory.

In retrospect, this problem of how to ensure the vanishing photon mass or the gauge symmetry in field theory was the starting point of the study of the quantum anomaly, the main subject of the present book. It is not obvious that one can maintain gauge symmetry or justify the unitary transformation to the interaction picture in field theory which deals with an infinite number of degrees of freedom. These are the essential aspects of symmetry breaking by the quantization procedure we are going to discuss.

1.3 The discovery of the quantum anomaly

We briefly explain the problem associated with the photon self-energy on the basis of relativistic quantum mechanics which can describe the creation and annihilation of particles in perturbation theory. A more detailed analysis will be given later in this book. We denote the interaction Hamiltonian, which describes the interaction between the photon (usually denoted by γ) and the electron (denoted by e) and the positron (denoted $\bar{e}$), by H_I. The photon self-energy in second-order perturbation theory is given by

$$\Delta E = \sum_n \langle\gamma|H_I|e\bar{e}\rangle \frac{1}{E - E_n} \langle e\bar{e}|H_I|\gamma\rangle. \tag{1.4}$$

The sum over the energy of the intermediate states $E_n = E_e + E_{\bar{e}}$ generally diverges. This is because the rapid increase in the number of states allowed for the electron–positron pair for increasing energy E_n and the allowed energy is unlimited for a relativistic theory in flat Minkowski space-time. This is a manifestation of the infinite number of degrees of freedom in momentum space for the creation of an electron–positron pair. The vanishing photon mass is ensured if the energy correction $\Delta E(\vec{k})$ as a function of the momentum $\vec{k}$ satisfies

$$\Delta E(\vec{0}) = 0. \tag{1.5}$$

In terms of the more intuitive Feynman diagram, this perturbation formula corresponds to a calculation of the Feynman diagram in Fig. 1.1. The photon, which is virtually converted to a pair of an electron and a positron, recombines after a certain time, and the quantum correction to the photon self-energy arises in this process. This diagram diverges (to be precise, a quadratic divergence), and the photon mass which should be zero may become indeterminate if the renormalization prescription should not be unique. As is explained later, the photon mass is in fact kept to be zero up to any finite order in perturbation in the modern formulation of quantum electrodynamics.

In 1949, however, this problem of the photon self-energy was the fundamental issue of renormalization theory, and the two members of Tomonaga's group,

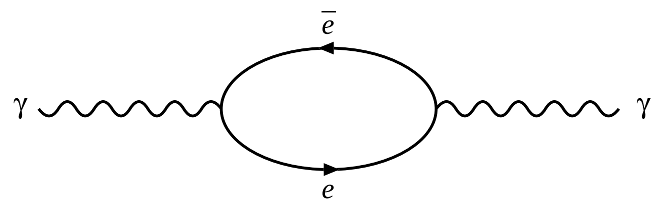

FIG. 1.1. Photon self-energy correction

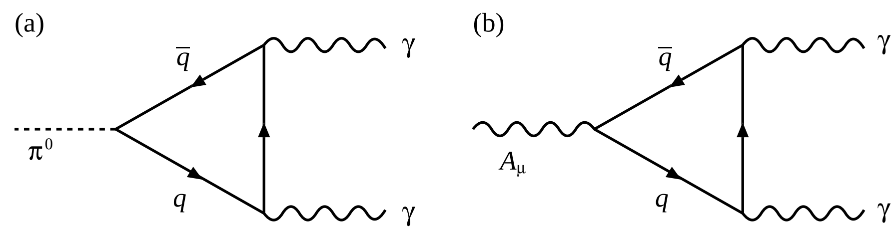

FIG. 1.2. (a) Two-photon decay of the neutral π meson; (b) two-photon decay of a neutral axial-vector meson

H. Fukuda and Y. Miyamoto, analyzed the next simplest Feynman diagrams in Fig. 1.2. (a) and (b) for the two-photon decay of the neutral π meson, which is commonly denoted by π^0, namely $\pi^0 \to \gamma\gamma$. In terms of modern language, they compared two processes: In one of them the neutral spinless π meson virtually splits into a quark q and anti-quark $\bar{q}$ pair in the vacuum, and those two quarks emit two photons before pair annihilation. In the other process, the spin 1 (axial-)vector meson A_μ dissociates into a quark q and anti-quark $\bar{q}$ pair and the pair of quarks emit two photons before pair annihilation. The prediction of renormalization theory is that there exists a simple relation (a symmetry called "Dyson's symmetry" at that time) between two processes. However, an explicit calculation indicated that the gauge invariance in the second graph is spoiled and that the relation between the two graphs does not hold. The latter graph diverges (to be precise a linear divergence) and thus a careful calculation is required. But the linear divergence already appeared in the self-energy of the electron in renormalization theory, and it is known that renormalization works without any difficulty for the electron self-energy. Consequently, the calculation of the triangle diagrams should also work.

The appearance of those discrepancies from the predictions of renormalization theory was a very serious issue for the renormalization program itself. In fact, Tomonaga, together with his collaborators, analyzed the triangle diagrams by using the Pauli–Villars regulator, of which a preprint was sent to Tomonaga from Pauli. They concluded that the issue of gauge invariance can be successfully handled by the Pauli–Villars regulator but the relation between the two graphs was not uniquely resolved. They stated that we have to wait for future experimental results to resolve the issue.

A similar and detailed analysis was performed by J. Steinberger at Princeton, who learned of the calculation of Fukuda and Miyamoto through Yukawa staying at Princeton at that time.[1] He also used the Pauli–Villars regulator and arrived at a conclusion similar to that of Tomonaga.

The issues related to the gauge invariance of the photon self-energy and the triangle diagrams were analyzed in greater detail by J. Schwinger in 1951. He handled the photon self-energy by showing that the gauge invariance can be consistently imposed but concluded anomalous behavior of the triangle diagrams. The distinction between the ambiguity related to divergences and the quantum breaking of symmetry was not clear at that time, and as a result a fundamental understanding of this strange behavior was not achieved.

This problem of the triangle diagrams for the pion (i.e., π meson) decay came up as a major issue again in the late 1960s. At that time, the understanding of the pion as a Nambu–Goldstone particle associated with spontaneous breaking of chiral symmetry (or PCAC) was established. If one combines this interpretation of the pion with the (naive) calculation of the triangle diagrams, it was concluded that the neutral pion cannot decay into two photons in the ideal limit of the Nambu–Goldstone particle. This contradicted the experimental fact that the neutral pion predominantly decays into two photons.

This difficulty was analyzed in great detail by J. Bell and R. Jackiw at CERN. Bell and Jackiw noticed the inevitable deviation from PCAC if one applies the conventional Pauli–Villars regularization to the σ model which incorporates PCAC. They then showed that one can preserve both PCAC and gauge invariance if one uses a modification of Gupta's implementation of the Pauli–Villars regularization, but this spoils renormalizability. On the other hand, S. Adler at Princeton performed a general analysis of the triangle diagrams in spinor electrodynamics and discussed the issue of the neutral pion decay in the appendix of his paper. The final conclusion was that a proper understanding of the anomalous behavior discussed by Fukuda, Miyamoto, Steinberger and Tomonaga resolves the discrepancy between theory and experiment. At the same time, it was concluded that the anomalous behavior of the triangle diagrams is unavoidable in relativistic local field theory with gauge symmetry. Namely, it was established that the Feynman diagrams can exhibit behavior different from a naive manipulation of canonical field theory and that the anomalous behavior is consistent with the basic postulate of local field theory and explains the experimental results well. In effect, these analyses in 1969 marked the discovery of the breaking of certain symmetries by the quantization procedure, namely, the quantum anomalies.

[1]Footnote 11 in J. Steinberger, *Phys. Rev.* **76** (1949) 1180, reads "Fukuda and Miyamoto, Prog. Theor. Phys. (in press), were the first to notice that the old results were not gauge invariant. Their work formed the starting point of this research. I wish to thank H. Yukawa for making their results available to me before publication". Incidentally, Steinberger later turned to experimental physics and received the Nobel Prize for the discovery of two neutrinos.

These papers influenced the entire subsequent developments of the subject. For example, motivated by these papers T. Kimura evaluated the triangle anomaly in the presence of the background gravitational field in the same year of 1969.

It is known that there are two main classes of symmetries which are broken by the quantization procedure. The first is the chiral symmetry associated with Dirac's γ_5 and it is related to the triangle diagrams we have discussed so far, and it is called the *chiral anomaly*. The other is the Weyl transformation, which changes the length scale of space-time, keeping the local angle invariant; this is called the *Weyl anomaly* or *conformal anomaly*.

On the other hand, in the formulation of renormalization theory Feynman invented the path integral methods of quantum mechanics and quantum field theory. The path integral and the canonical operator formulation of quantum theory are formally equivalent, but the path integral later found many applications in practical calculations. Combined with an intuitive understanding of quantum processes, the path integral is becoming increasingly important in the modern formulation of quantum theory and, in particular, quantum field theory.

It has been recognized that the quantum anomalies are understood as arising from non-trivial Jacobians associated with the change of integration variables in the path integral formulation. The path integral measure breaks those symmetries. This path integral method provides a conceptually more satisfactory derivation of anomalous identities with the anomaly terms present from the beginning instead of discovering the anomalies after the evaluation of current divergences. The purpose of this book is to provide an introduction to the path integral method of quantization and its applications to the analyses of quantum anomalies. We show that quantum anomalies are phenomena basic to the entire field theory, in particular, to gauge theory in general on the basis of the path integral formulation. We thus start with an introductory account of the path integral formulation of quantum field theory in the next chapter.