Exercises and solutions for 'Renormalization methods: a guide for beginners'

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Contents

Part I

Exercises

A Revision of statistical physics based on Appendices A and B

1. The eigenvalues of the Schrödinger equation for an assembly of N independent particles in a volume V take the form:
$$E_i = B_i V^{-\gamma}$$
where the B_i are independent of V, and $\gamma = 2/3$ or $1/3$, according to whether the particles have energies which are small or large when compared with their rest mass energy. Show that the pressure in such an assembly is given by
$$P = \gamma \overline{E} V^{-1}.$$

2. Show that the fluctuations in the pressure and total energy of a fluid in a container of fixed volume V, satisfy the relation
$$\langle \Delta E \Delta P \rangle = kT^2 \left(\frac{\partial \langle P \rangle}{\partial T} \right)_V,$$
provided that the fluid is in thermal equilibrium.

3. Consider a fluid in a container whose walls are flexible so that its volume may take on any one of a set of values $\{V_\alpha\}$. Obtain the probability $p_{i,\alpha}$ of finding the fluid to be in the eigenstate i with volume V_α. Identify the two Lagrange multipliers appearing in your result, given that the mean energy and mean volume of the assembly are $\overline{E}$ and $\overline{V}$, respectively.

4. If the Gibbs entropy, as given by eqn (A.8) for an assembly in an ensemble, is specialised to the case of an isolated system, show that it reduces to the usual Boltzmann form, viz.,
$$S = k \ln \Omega,$$
where Ω is the statistical weight of a given macrostate.

5. A system in the canonical ensemble has access to states with energy eigenvalues E_i. Show that its probability of occupying a state $\mid i \rangle$ can be written as
$$p_i = e^{\beta(F - E_i)},$$
where $\beta = 1/kT$ and F is the Helmholtz free energy.

Verify that the mean energy of the assembly can be written as
$$\overline{E} = \frac{\partial}{\partial \beta}(\beta F),$$
and further show that the general $(n+1)$-order moment of the deviation from its mean value satisfies
$$\langle (E - \overline{E})^{n+1} \rangle = -\frac{\partial}{\partial \beta} \langle (E - \overline{E})^n \rangle + n \langle (E - \overline{E})^{n-1} \rangle \frac{\partial \overline{E}}{\partial \beta}.$$
By specialising this general form, obtain the result
$$C_V = \frac{1}{kT^2} \left[\langle E^2 \rangle - \langle E \rangle^2 \right].$$

B Introducing magnetism, fluctuations and critical phenomena

1. A magnetic assembly at a fixed temperature T is subject to an applied magnetic field B. If the net magnetic moment of the assembly in state $|i\rangle$ is M_i, in the direction of the applied field, then the associated energy eigenvalue is given by

$$E_i = e_i(B=0) - M_i B,$$

where $e_i(B=0)$ is the energy arising from the mutual interaction of lattice spins.

Show that fluctuations of the magnetic moment about its mean value are given by

$$\triangle M^2 = \langle M^2 \rangle - \langle M \rangle^2 = kT\chi_T,$$

where χ_T is the isothermal magnetic susceptibility defined by

$$\chi_T = [\partial \langle M \rangle / \partial B]_T.$$

2. Show that the isothermal compressibility of an ideal gas is given by

$$K_T^o = \frac{1}{nkT}$$

where n is the number density

[Note: in general the isothermal compressibility of a fluid is given by

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T .]$$

3. If the number of particles N in an assembly fluctuates about some mean value $\overline{N}$, the partition function may be written as

$$\mathcal{Z} = \sum_{i,N} e^{-\beta E_i + \beta \mu N},$$

where μ is the chemical potential and the other symbols have their usual meaning. Verify by direct differentiation that the mean-square fluctuation in particle number is given by:

$$\langle N \rangle^2 - \langle N^2 \rangle = (kT)^2 \left.\frac{\partial^2 \ln Z}{\partial \mu^2}\right)_{T,V} .$$

4. Show that the mean-square fluctuation in particle number in a fluid may be expressed in terms of the isothermal compressibility K_T, Thus:

$$\frac{K_T}{K_T^0} = \frac{\langle (N - \langle N \rangle)^2 \rangle}{\langle N \rangle}$$

where K_T^0 is the compressibility of an ideal gas.

[Note: the following relationships may be found helpful:

$$PV = kT \ln \mathcal{Z},$$

$$\left(\frac{\partial P}{\partial \mu}\right)_{T,V} = n = \frac{\overline{N}}{V},$$

and

$$K_T = -\frac{1}{\overline{N}}\left(\frac{\partial V}{\partial \mu}\right)_{T,N},$$

where μ is the chemical potential]

5. Sketch the isotherms for a fluid in the PV plane for T in the neighbourhood of T_c. Comment on the behaviour of the isothermal compressibility as $T \to T_c$, and also on the related behaviour of the density fluctuations.

   Show that a divergent susceptibility is mathematically related to an increase in the correlation length of density-density fluctuations and comment on the experimental manifestation of this effect.

   [Note: the density-density correlation $G(\mathbf{r} - \mathbf{r}')$ is defined by:

$$G(\mathbf{r} - \mathbf{r}') \equiv \langle \{n(\mathbf{r}) - \langle n(\mathbf{r})\rangle\}\{n(\mathbf{r}') - \langle n(\mathbf{r}')\rangle\}\rangle,$$

   where $n(\mathbf{r})$ is the local number density].

C Exercises for Chapter One

1. Show that the number a is a fixed point of the dynamical system

$$X(n+1) = f((X(n)),$$

   if a satisfies the equation:

$$a = f(a).$$

   Hence show:

   (a) That $a = 3$ is the fixed point of the dynamical system

$$X(n+1) = 2X(n) - 3.$$

   (b) That the dynamical system

$$X(n+1) = rX(a) + b$$

   only has a fixed point at infinity if $r = 1$.

2. Obtain and verify the two fixed points of the dynamical system

$$X(n+1) = [X(n) + 4]X(n) + 2.$$

   Calculate $X(n)$ to four-figure accuracy for initial values $X(0) = -1.01, -0.99$ and -2.4 taking values of n up to $n = 7$ and comment on the results.

3. The **logistic equation** of population growth takes the form

$$X(n+1) = (1+r)X(n) - bX^2(n)$$

   where r is the growth rate (r = births – deaths) and $b = r/L$, where L is the maximum population which the environment can support. Obtain the two fixed-point values of the system and briefly comment on their physical significance.

4. For any system which undergoes a single phase transition, two of the fixed points may immediately be identified as the low-temperature and high-temperature fixed points. Discuss the physical significance of these points and explain why they are attractive.

   How would your conclusions be affected if we chose to apply these arguments to the one-dimensional Ising model as a special case?

5. A system has energy levels $E_i = 0, \epsilon_1, \epsilon_2, \epsilon_3 \ldots$ with degeneracies $g_i = 1, 2, 2, 1 \ldots$. The system is in equilibrium with a thermal reservoir at temperature T, such that $e^{-\beta\epsilon_j} \ll 1$ for $j > 4$. Work out the partition function, the mean energy and the mean squared energy of the system.

   N.B. An energy level with a degeneracy of two implies two states with the same energy. The partition function is a sum over *states*, not *levels*.

6. When a particle with spin $\frac{1}{2}$ is placed in a magnetic field B, its energy level is split into $\pm\mu B$ and it has a magnetic moment $\pm\mu$ (respectively) along the direction of the magnetic field. Suppose that an assembly of N such particles on a lattice is placed in a magnetic field B and is kept at a temperature T. Find the internal energy, the entropy, the specific heat and the total magnetic moment of this assembly.

D Exercises for Chapter Two

1. A linear chain Ising model has a Hamiltonian given by

$$H = -\sum_{i=1}^{N-1} J_i S_i S_{i+1}.$$

Show that the partition function takes the form:

$$Z_N = 2^N \prod_{i=1}^{N-1} \cosh K_i,$$

where $K_i \equiv J_i/kT$ is the coupling parameter.

Obtain the spin-spin correlation function in terms of the partition function and show that for uniform interaction strength it takes the form:

$$G_n(r) \equiv \langle S_n S_{n+r} \rangle = \tanh^r K.$$

where K is the coupling constant for the model.

Comment on the possibility of long-range order appearing in the cases: (a) $T > 0$; and (b) $T = 0$.

2. Show that the transfer matrix method may be extended to the case of an Ising ring located in a constant field B and hence obtain the free energy per lattice site in the form:

$$f = -J - \frac{1}{\beta} \ln[\cosh \beta B \pm (e^{2\beta J} \sinh^2 \beta B + e^{2\beta J})^{\frac{1}{2}}],$$

where all the symbols have their usual meaning.

[Hint: When generalizing the transfer matrix to the case of non-zero external field remember that it must remain symmetric in its indices.]

Obtain an expression for the specific magnetization $m = \langle S \rangle$, and discuss its dependence on the external field B. comment on your results.

3. The RG recursion relations for a one-dimensional Ising model in an external field B may be written as;

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)}; \quad y' = \frac{y(x+y)}{(1+xy)},$$

where $x = e^{-4J/kT}$ and $y = e^{-B/kT}$. Verify the existence of fixed points as follows: $(x^*, y^*) = (0, 0)$; $(x^*, y^*) = (0, 1)$ and a line of fixed points $x^* = 1$ for $0 \leq y^* \leq 1$. Discuss the physical significance of these points and sketch the system point flows in the two-dimensional parameter space bounded by $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

4. Verify that summing over alternate spins on a square lattice results in a new square lattice rotated through 45° relative to the original lattice and with a scale factor of $b = \sqrt{2}$.

Also verify that the effect of such a decimation on the Ising model is to change the original partition function involving only pairs of nearest neighbours to a form involving nearest-neighbours, next-to-nearest neighbours and the product of four spins taken 'round a square'.

E Exercises for Chapter Three

1. On the basis of the Weiss molecular field theory, show that the Ising model cannot exhibit a spontaneous magnetisation for temperatures greater than $T = T_c$, where the Curie temperature is given by $T_c = J/k$.

2. An isolated parallel plate capacitor has a potential difference V between its electrodes, which are situated at $x = \pm a$. The space between the electrodes is occupied by an ionic solution which has a dielectric constant of unity. Obtain an expression for $n(x)$, the space charge distribution, which exists after the system has reached thermal equilibrium. For the sake of simplicity you may assume that the potential difference between the plates is so small that $eV \ll kT$.

   [Hint: take $x = 0$ as a plane of symmetry where:

   (a) $n_+(0) = n_-(0) = n_0$ (say)

   (b) the potential equals zero.]

3. Using the Landau model for phase transitions obtain values for the critical exponents β, γ and δ.

F Exercises for Chapter Four

1. Use the high-temperature expansion to show that the partition function of the (open) Ising linear chain takes the form
$$Z_N = 2^N \cosh^{N-1} K,$$
and explain the differences between this and the result for the Ising ring.

2. Use the method of high-temperature expansion to obtain the spin-spin correlation $\langle S_n S_m \rangle$ of the Ising model as
$$\langle S_m S_n \rangle = Z_N^{-1} \cosh^P K \, 2^N \sum_{r=1}^{P} f_{mn}(r) v^r,$$
where P is the number of nearest neighbour pairs, $v \equiv \tanh K$ and $f_{mn}(r)$ is the number of graphs of r lines, with even vertices except at sites m and n.

   Show that for the case of the Ising linear chain, this reduces to
$$\langle S_m S_n \rangle = v^{|n-m|},$$
and draw the corresponding graph.

3. If one can assume that the interparticle potential $\phi(r)$ is large (and positive) on the scale of kT, for $r < d$; and is small for $r > d$, where d is the molecular diameter, then it may be shown that the second virial coefficient can be written as
$$B_2 = B - A/kT.$$
Obtain explicit expressions for the constants A and B, and show that the resulting equation of state may be reduced to the Van der Waal's equation.

4. If a gas of interacting particles is modelled as hard spheres of radius a, show that the second virial coefficient takes the form:
$$B_2 = \frac{2\pi a^3}{3}.$$
Given that the third virial coefficient may be written as:
$$B_3 = \frac{1}{3} \int d^3 r \int d^3 r' f(|\mathbf{r}|) f(|\mathbf{r}'|) f(|\mathbf{r} - \mathbf{r}'|),$$

where $f(r) = e^{-\beta\Phi(r)} - 1$, show that this is related to the second virial coefficient B_2 by

$$B_3 = \frac{5}{8} B_2^2 ,$$

for a system of hard spheres.

5. A gas consisting of N classical point particles of mass m occupies a volume V at temperature T. If the particles interact through a two-body potential of the form:

$$\phi(r_{ij}) = \frac{A}{r_{ij}^n} ,$$

where A is a constant, $r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$ and n is positive, show that the canonical partition function is a homogeneous function, in the sense

$$Z(\lambda T, \lambda^{-3/n} V) = \lambda^{3N(\frac{1}{2}-\frac{1}{n})} Z(T, V) ,$$

where λ is an arbitrary scaling factor.

6. Prove the identity

$$\frac{\partial}{\partial x} e^{-\beta H} = -\int_0^\beta e^{-(\beta-y)H} \frac{\partial H}{\partial x} e^{-yH} dy,$$

to second order in β by equating coefficients in the high-temperature (small β) expansion of each side of the relation, where H is an operator.

7. Show that at high temperatures, the heat capacity of a quantum assembly can be written as

$$C_V = \frac{1}{kT^2} \left\{ \frac{Tr(H^2)}{Tr(1)} - \frac{[Tr(H)]^2}{[Tr(1)]^2} + 0(\beta) \right\} ,$$

where $\beta = 1/\mathrm{kT}$.

8. Show that the use of the Van der Waals equation,

$$\left(P + \frac{a}{V^2}\right)(V - b) = NkT,$$

to describe phase transitions in a fluid system leads to the following values for the critical parameters:

$$P_c = a/27b^2,\ V_c = 3b,\ NkT_c = 8a/27b.$$

Hence show that the Van der Waals equation may be written in the universal form

$$\left(\tilde{p} + \frac{3}{\tilde{v}^2}\right)(3\tilde{v} - 1) = 8\tilde{t},$$

where

$$\tilde{p} = P/P_c,\ \tilde{v} = V/V_c,\ \tilde{t} = T/T_c.$$

9. By re-expressing the Van der Waals equation in terms of the reduced variables; $P = (P - P_c)/P_c$, $v = (V - V_c)/V_c$ and $\theta_c = (T - T_c)/T_c$, obtain values for the critical exponents γ and δ. Comment on the values which you obtain.

G Exercises for Chapter Seven

1. Prove the critical exponent inequality $\alpha + 2\beta + \gamma \geq 2$

   [Hint: you may assume the relationship

$$\chi_T(C_B - C_M) = T(\partial M/\partial T)_B^2,$$

   where C_B and C_M are the specific heats at constant field and magnetisation respectively.

2. Additional critical exponents can be obtained if we differentiate the Gibbs free energy repeatedly with respect to the external magnetic field B, thus:

$$(\partial^\ell G/\partial B^\ell)_T = G^{(\ell)} \sim \theta_C^{\triangle_\ell} G^{(\ell-1)},$$

   where the $\triangle_\ell$ are known as the "gap exponents" and θ_c is the reduced temperature. On the basis of the Widom scaling hypothesis, show that the gap exponents are all equal and give their value in terms of the parameters of the Widom scaling transformation.

3. If we denote the order parameter (or specific magnetisation) by M, show that the mean-field solution of the Ising model can be written as

$$M = \tanh\left[\frac{M}{1+\theta_c} + b\right],$$

   where θ_c is the reduced temperature and $b = \beta B$ is the reduced external magnetic field.

   By considering m for temperature close to T_c and for zero external field, show that the associated critical exponent takes the value $\beta = \frac{1}{2}$.

   [Hint: the following expansion

$$\tanh x = x - \frac{1}{3}x^3 + 0(x^5),$$

   for small values of x, should be helpful.]

4. Obtain an expression for the mean energy $\overline{E}$ of the Ising model when the applied field is zero, using the simplest mean-field approximation. Hence show that the specific heat C_B has the behaviour:

$$\begin{aligned} C_B &= 0, && \text{for} T > T_c; \\ &= 3Nk/2, && \text{for} T < T_c. \end{aligned}$$

   What is the value of the associated critical exponent α ?

5. By considering the behaviour of the order parameter at temperatures just above the critical point, show that the critical exponent γ, which is associated with the isothermal susceptibility, takes the value $\gamma = -1$ according to mean-field theory.

   Also, by considering the effect of an externally applied magnetic field at $T = T_c$, show that the exponent associated with the critical isotherm takes the value $\delta = 3$. [In the latter case, the identity

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y},$$

   may be helpful.]

6. Show that the optimal free energy of an Ising model, which corresponds to the mean-field theory, may be written in the form:

$$F = -\frac{N}{\beta}\ln[2\cosh(\beta B_E)] + \frac{N}{2zJ}(B_E - B)^2,$$

   where $\beta = 1/kT$, B is the externally applied magnetic field, B_E is the effective field experienced by each spin, z is the coordination number and J is the interaction strength.

7. Consider an Ising model where the external field B_i depends on the position of the lattice site i. Show that the condition for the free energy to be a minimum takes the form:

$$B_E^{(i)} - B_i = J \sum_{\langle j \rangle} \langle S_j \rangle_o \,,$$

where all the symbols have their usual meaning and the notation indicates that the sum over j is restricted to nearest neighbours of i. Also show that the optimal free energy takes the form:

$$F = \frac{-N}{\beta} \ln \left[2 \cosh \left(\beta B_E^{(i)} \right) \right] + \frac{1}{2} \sum_i \left(B_E^{(i)} - B_i \right) \langle S_i \rangle_o \,.$$

8. A generalized Ising model has the usual Hamiltonian but each spin variable takes the values:

$$S_i = -t, -t+1, \ldots., t-1, t,$$

where t may be either an integer or a half-odd integer. Using mean-field theory find the critical temperature of this system, and then use this result to recover the critical temperature corresponding to the standard two-state Ising model.

[Note: you may assume the relationship:

$$\sum_{S=-t}^{t} e^{xS} = \frac{\sinh \left[\left(t + \frac{1}{2} \right) x \right]}{\sinh[x/2]} \,.]$$

9. The Heisenberg model for ferromagnetism is given by the Hamiltonian:

$$H = -J \sum_{<i,j>} \mathbf{S}_i . \mathbf{S}_j - \sum_i \mathbf{B} . \mathbf{S}_i \,,$$

where $\mathbf{S}_i$ is a three-dimensional unit vector and $\mathbf{B}$ is a uniform external magnetic field. Use mean field theory to obtain an expression for the critical temperature T_c.

[Hint: Take the external field to be in the $\hat{z}$ direction.]

10. The Hamiltonian of a certain model system is given by

$$H = \frac{-J}{N} \sum_{ij} S_i S_j - B \sum_i S_i \,,$$

with $S_i = \pm 1$. Show that the system undergoes a phase transition in all dimensions and find the critical temperature in the thermodynamic limit $N \to \infty$.

Comment on the implications of this result for the Ising model.

H Exercises for Chapter Eight

1. The RG recursion relations for a one-dimensional Ising model in an external field B may be written as;

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)}; \quad y' = \frac{y(x+y)}{(1+xy)},$$

where $x = e^{-4J/kT}$ and $y = e^{-B/kT}$. Verify the existence of fixed points as follows: $(x^*, y^*) = (0,0)$; $(x^*, y^*) = (0,1)$ and a line of fixed points $x^* = 1$ for $0 \le y^* \le 1$. Discuss the physical significance of these points and sketch the system point flows in the two-dimensional parameter space bounded by $0 \le x \le 1$ and $0 \le y \le 1$. By linearising about the ferromagnetic fixed point, obtain the matrix of the RGT and show that the associated critical indices are $y_1 = 2, y_2 = 1$ for scaling factor $b = 2$.

2. The Hamiltonian for the one-dimensional Ising model in an external field B may be written as

$$H = -K \sum_{<ij>} S_i S_j - -\beta B \sum_i S_i - \sum_i C,$$

where C is a constant, background term. Show that the partition function for the system may be written as a product of terms, each of which depends on only one of the even-numbered spins. Hence, by partial summation over the even-numbered spins, obtain the recursion relations for K', B' and C'.

3. Consider the two-dimensional Ising model under decimation on a square lattice. If we only take into account the coupling constants K (nearest-neighbour interactions) and L (next-nearest-neighbour interactions), the recursion relations are given by

$$K' = 2K^2 + L; \quad L' = K^2.$$

Find the fixed points for these renormalization group equations and identify the critical one. Linearising the RGT about this point, obtain a value for the critical exponent ν.

4. Discuss bond percolation on a two-dimensional square lattice and, drawing analogies with the Ising model where appropriate, introduce the concepts of *critical probability, correlation length,* and *critical exponent.*

   Show that the critical probability for this problem is $p_c = \frac{1}{2}$.

   Apply the renormalization group to bond percolation, using a scheme in which alternate sites are removed from the lattice. Show that the RG equation takes the form

$$p' = 2p^2 - p^4,$$

   where p' is the probability of two sites being connected on the new lattice and p is the analogous quantity for the old lattice. Verify that the fixed points are $p^* = 0, p^* = 1$, and $p^* = (\sqrt{5}-1)/2 \simeq 0.62$, and discuss the nature of these fixed points.

   By linearizing the RG transformation about a fixed point, show that the relevant eigenvalue of the transformation matrix is given bu

$$\lambda_1 = 4p^*(1 - p^{*2}).$$

   Using this result, obtain a numerical value for the critical exponent ν.

I Exercises for Chapter Nine

1. A particular model for the critical behaviour of spins on a d-dimensional lattice leads to renormalization group equations of the form

$$p' = b^2 p + C(b^2 - b^\epsilon)q + O(pq);$$

$$q' = b^\epsilon q\,,$$

   where p and q are the coupling constants, C is a system constant which is positive and real, $\epsilon = 4 - d$, and b is the usual (length) scaling factor. By linearising about the fixed point $(p^*, q^*) = (0, 0)$, obtain the matrix of the RGT and show that the associated critical indices are $y_1 = 2$ and $y_2 = 4 - d$.

   Briefly discuss the nature of this fixed point.

   Given that the critical indices for this model are the same as those for the mean-field theory of the Ising model, comment briefly on the validity or otherwise of mean-field theory.

2. A particular model for the critical behaviour of spins on a d-dimensional lattice leads to renormalization group equations of the form

$$\frac{d\mu(b)}{d\ln b} = 2\mu(b) + \frac{\Lambda^2\lambda(b)}{16\pi^2} - \frac{\mu(b)\lambda(b)}{16\pi^2};$$

$$\frac{d\lambda(b)}{d\ln b} = \epsilon\lambda(b) - \frac{3\lambda^2(b)}{16\pi^2},$$

where μ and λ are the coupling constants, Λ is a system constant which is positive and real, $\epsilon = 4 - d$, and b is the usual spatial rescaling factor. These equations are valid for small values of ϵ. Given that the criterion for a fixed point (μ^*, λ^*) is

$$\frac{d\mu^*(b)}{d\ln b} = \frac{d\lambda^*(b)}{d\ln b} = 0,$$

verify that a fixed point (to order ϵ) is given by

$$\lambda^* = \frac{16\pi^2\epsilon}{3}; \qquad \text{and} \qquad u^* = -\frac{\Lambda^2\epsilon}{6}.$$

By linearising about the fixed point, obtain the matrix of the RGT and show that the associated critical indices are $y_1 = 2 - \epsilon/3$ and $y_2 = -\epsilon$.

Briefly discuss the nature of the fixed point with particular reference to the dimensionality of the lattice.

Show that an expression for the critical exponent ν can be written as:

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} + \mathcal{O}(\epsilon^2).$$

Part II
Solutions

J Solutions to Exercises A

1. The analogue of eqn (B.11) for pressure in the canonical ensemble when extended to the grand canonical ensemble is

$$P = -\sum_{i,N}\left(\frac{\partial E_{i,N}}{\partial V}\right)p_{i,N}.$$

Given:

$$E_{i,N} = B_i V^{-\gamma}$$

therefore

$$P = -\sum_{i,N}(-\gamma)B_iV^{-\gamma-1}p_{i,N} = \gamma\sum_{i,N}\frac{B_iV^{-\gamma}}{V}dp_{I,N} = \frac{\gamma}{V}\sum_{i,N}E_{i,N}p_{i,N} = \frac{\gamma\overline{E}}{V}.$$

2.

$$\langle\Delta E\cdot\Delta P\rangle = \langle(\overline{E}-E)(\overline{P}-P)\rangle = \langle E\cdot P\rangle - \langle E\rangle\langle P\rangle.$$

Now (B.13) gives

$$P_i = -\partial E_i/\partial V,$$

hence

$$\langle E\cdot P\rangle = \sum_i p_iE_i\frac{\partial E_i}{\partial V} = -\frac{1}{\mathcal{Z}}\sum_i e^{-E_i/kT}\cdot E_i\frac{\partial E_i}{\partial V}.$$

Clearly we now want to get this into the form:

$\langle E\rangle\langle P\rangle$ plus 'something else'.

Note:

$$E_i e^{-E_i/kT} = (+kT^2)\frac{\partial}{\partial T}e^{-E_i/kT} = kT^2\frac{\partial}{\partial T}(\mathcal{Z}p_i).$$

Substitute into the formula for $\langle E\cdot P\rangle$ to obtain:

$$\langle E\cdot P\rangle = -\frac{1}{\mathcal{Z}}kT^2\frac{\partial}{\partial T}\mathcal{Z}\sum_i p_i\partial E_i/\partial V = \frac{kT^2}{\mathcal{Z}}\frac{\partial}{\partial T}\{\mathcal{Z}\langle P\rangle\},$$

(remember $P_i = -\partial E_i/\partial V$).

Now differentiate the product $\mathcal{Z}\langle P\rangle$ with respect to T and use the identity:

$$\mathcal{Z}^{-1}\partial\mathcal{Z}/\partial T = \partial\ln\mathcal{Z}/\partial T,$$

with the result:

$$\langle E\cdot P\rangle - \langle P\rangle\langle E\rangle = \langle\Delta E\cdot\Delta P\rangle = kT^2\partial\langle P\rangle/\partial T.$$

3. Put $\lambda_y = \lambda_E$ and $\lambda_z = \lambda_\alpha$: then eqn (A.23) becomes

$$p_{i,\alpha} = \mathcal{Z}^{-1}e^{-(\lambda_E E_{i,\alpha}+\lambda_\alpha V_\alpha)},$$

and (A.24), (A.25) can be generalised as appropriate. The constraints are given by

$$\overline{E} = \sum_{i,\alpha}p_{i,\alpha}E_{i,\alpha};\qquad \overline{V} = \sum_{i,\alpha}p_{i,\alpha}V_\alpha.$$

To identify the multipliers $\lambda_E, \lambda_\alpha$ we compare the macroscopic expression for $d\overline{E}$ (combined 1st and 2nd laws of thermodynamics) with the microscopic prediction.

From thermodynamics $$d\overline{E} = TdS - PdV + \sum_\gamma X_\gamma dx_\gamma.$$

From eqn$(B.6)$ $$d\overline{E} = \sum_{i,\alpha}E_{i,\alpha}dp_{i,\alpha} + \sum_{i,\alpha}p_{i,\alpha}dE_{i,\alpha}.$$

Also, $$dS = \lambda_E\sum_{i,\alpha}E_{i,\alpha}dp_{i,\alpha} + \lambda_\alpha\sum_{i,\alpha}V_\alpha dp_{i,\alpha}.$$

Entropy is varied at constant V, so the constraint on $\overline{V}$ gives

$$d\overline{V} = \sum_{i,\alpha}V_\alpha dp_{i,\alpha},$$

thus

$$dS = \lambda_E\sum_{i,\alpha}E_{i,\alpha}dp_{i,\alpha} + \lambda_\alpha d\overline{V},$$

and so

$$\sum_{i,\alpha}E_{i,\alpha}dp_{i,\alpha} = \frac{dS}{\lambda_E} - \frac{\lambda_\alpha}{\lambda_E}d\overline{V},$$

hence

$$d\overline{E} = \frac{ds}{\lambda_E} - \frac{\lambda_\alpha}{\lambda_E}d\overline{V} + \sum_\gamma\left\{\sum_{i,\alpha}p_{i,\alpha}\frac{\partial E_{i,\alpha}}{\partial x_\gamma}dx_\gamma\right\}.$$

Comparison of the 'micro' and 'macro' forms of $d\overline{E}$ leads to

$$\lambda_E = 1/T;\qquad \lambda_\alpha = P/T.$$

4. By equal *a priori* probabilities

$$p_i = \frac{1}{\Omega},$$

for all i, where Ω is the number of microstates. The Gibbs entropy is given by

$$S = -k \sum_i p_i \ln p_i.$$

Every member of the summation is the same and there are Ω microstates, hence

$$S = -k\left(\Omega \times \frac{1}{\Omega}\right)\left(\ln \frac{1}{\Omega}\right) = -k(-\ln \Omega) = k \ln \Omega,$$

as required.

5. From (B.18) we have

$$\overline{E} = -k \frac{\partial}{\partial(1/T)} \ln \mathcal{Z} = -\frac{\partial \ln \mathcal{Z}}{\partial \beta},$$

for $\beta = 1/kT$.
From thermodynamics, free energy $F = \overline{E} - TS$ and

$$S = -k \sum_i p_i \ln p_i = k \ln \mathcal{Z} + \overline{E/T}.$$

Thus:

$$F = -kT \ln \mathcal{Z} = -\beta^{-1} \ln \mathcal{Z}.$$

It follows that

$$\mathcal{Z} = e^{-\beta F}$$

and,

$$p_i = e^{\beta(F - E_i)}.$$

and so:

$$\overline{E} = -\frac{\partial}{\partial \beta}(-\beta F) = \frac{\partial}{\partial \beta}(\beta F).$$

Now

$$\begin{aligned}
\langle \quad & (E - \overline{E})^{n+1}\rangle = \sum_i (E_i - \overline{E})^{n+1} p_i \\
= \quad & \sum_i (E_i - \overline{E})^{n+1} e^{\beta(F - E_i)} = -\sum (E_i - \overline{E})^n \frac{\partial}{\partial \beta} e^{\beta(F - E_i)} \\
= \quad & -\frac{\partial}{\partial \beta} \sum_i (E_i - \overline{E})^n e^{\beta(F - E_i)} + \sum_i e^{\beta(F - E_i)} \frac{\partial}{\partial \beta}(E_i - \overline{E})^n \\
= \quad & -\frac{\partial}{\partial \beta}\langle (E - \overline{E})^n\rangle + n \sum_i e^{\beta(F - E_i)} (E - \overline{E})^{n-1} \frac{\partial \overline{E}}{\partial \beta} \\
= \quad & -\frac{\partial}{\partial \beta}\langle (E - \overline{E})^n\rangle + n\langle (E - \overline{E})^{n-1}\rangle \frac{\partial \overline{E}}{\partial \beta}.
\end{aligned}$$

For the case $n = 1$:

$$\langle \Delta E^2\rangle = \langle (E - \overline{E})^2\rangle$$

$$\langle E - \overline{E}\rangle = 0.$$

Making the change of variable and doing the differentiation at constant V,

$$\frac{\partial \overline{E}}{\partial \beta} \to kT^2 C_V.$$

K Solutions to Exercises B

1. Given

$$E_i = e_i(B=0) - M_i B, \qquad (*)$$

the probability of assembly being in state $|i\rangle$ is

$$p_i = \frac{e^{-E_i/kT}}{\mathcal{Z}}.$$

Obviously from equation (*), the instantaneous magnetic moment is given by

$$M_i = -\partial E_i/\partial B.$$

Thus mean value of magnetic moment is given by:

$$\langle M\rangle = \sum_i M_i p_i = -\sum_i \frac{\partial E_i}{\partial B}\cdot\frac{e^{-E_i/kT}}{\mathcal{Z}} = -kT\frac{\partial \ln\mathcal{Z}}{\partial B}.$$

Introduce:

$$\begin{aligned}\chi_T &= \left.\frac{\partial\langle M\rangle}{\partial B}\right)_T = \sum_i M_i \frac{\partial p_i}{\partial B} \\ &= \frac{-1}{kT}\sum_i M_i p_i \frac{\partial E_i}{\partial B} - \frac{\partial \ln\mathcal{Z}}{\partial B}\sum_i M_i p_i \\ &= \frac{1}{kT}\{\langle M^2\rangle - \langle M\rangle^2\},\end{aligned}$$

therefore

$$\Delta M^2 = \langle M^2\rangle - \langle M\rangle^2 = kT\chi_T.$$

2.

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T.$$

For an ideal gas,

$$PV = NkT$$

(or $P = \frac{N}{V}kT = nkT$)
thus

$$V = NkT/P$$

and so

$$K_T^0 = -\frac{1}{V}\left(\frac{-NkT}{P^2}\right) = \frac{1}{V}\cdot\frac{1}{P^2}\cdot PV = \frac{1}{P} = \frac{1}{nkT} \quad \text{(using above equation in brackets)}$$

hence

$$K_T^0 = \frac{1}{nkT},$$

as required.

3. Mean-square fluctuation:

$$\begin{aligned}\langle(N-\langle N\rangle)^2\rangle &= \langle N^2 - 2N\langle N\rangle + \langle N\rangle^2 > \\ &= \langle N^2\rangle - 2\langle N\rangle^2 + \langle N\rangle^2 \\ &= \langle N^2\rangle - \langle N\rangle^2 \\ &\equiv \langle N^2\rangle - \overline{N}^2.\end{aligned}$$

Given: $\mathcal{Z} = \sum_{i,N} e^{-\beta E_i + \mu N\beta}$

$$\begin{aligned}
(kT)^2 \left.\frac{\partial^2 \ln \mathcal{Z}}{\partial \mu^2}\right)_{T,V} &= (kT)^2 \frac{\partial}{\partial \mu}\left[\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \mu}\right]_{T,V} \\
&= (kT)\frac{\partial}{\partial \mu}\left[\frac{1}{\mathcal{Z}}\sum_{i,N} e^{-\beta E_i + \beta\mu N} \times N\right] \\
&= kT\left[-\frac{1}{\mathcal{Z}^2}\frac{\partial \mathcal{Z}}{\partial \mu}\cdot\sum_{i,N} N e^{-\beta E_i + \beta\mu N}\right. \\
&+ \left.\frac{1}{\mathcal{Z}}\sum_{i,N}\beta N^2 e^{-\beta E_i + \beta\mu N}\right] \\
&= [-\langle N\rangle^2 + \langle N^2\rangle] \\
&= \langle N^2\rangle - \langle N\rangle^2,
\end{aligned}$$

as required.

4. From the previous question:

$$\langle (N - \langle N\rangle)^2\rangle = (kT)^2\left(\frac{\partial^2 \ln \mathcal{Z}}{\partial \mu^2}\right) T, V.$$

Now $\mathcal{Z}$ is partition function for the Grand Canonical ensemble, hence we have:

$$PV = kT \ln \mathcal{Z}.$$

Substituting for $\ln \mathcal{Z}$, we have:

$$\begin{aligned}
(kT)^2 \left.\frac{\partial^2 \ln \mathcal{Z}}{\partial \mu^2}\right)_{T,V} &= (kT)^2\left(\frac{\partial^2}{\partial \mu^2}\left[\frac{PV}{kT}\right]\right)_{T,V} \\
= (kT)^2 \frac{V}{kT}\left.\frac{\partial^2 P}{\partial \mu^2}\right)_{T,V} &= kTV\frac{\partial}{\partial \mu}\left(\frac{\partial P}{\partial \mu}\right) T, V.
\end{aligned}$$

Given:

$$\left(\frac{\partial P}{\partial \mu}\right)_{T,V} = \frac{\overline{N}}{V} = n,$$

$$\begin{aligned}
\Rightarrow\ kTV\frac{\partial}{\partial \mu}\, n)_V = kTV \left.\frac{\partial n}{\partial \mu}\right)_N &= kTV\left(\frac{\partial n}{\partial V}\right)\left(\frac{\partial V}{\partial \mu}\right)_N \\
= \ kTV\left(\frac{\partial}{\partial V}\cdot\frac{\overline{N}}{V}\right)\left(\frac{\partial V}{\partial \mu}\right)_N &= \frac{-kTV\overline{N}}{V^2}\left(\frac{\partial V}{\partial \mu}\right)_N \\
= \ \frac{-kT\overline{N}}{V}\left(\frac{\partial V}{\partial \mu}\right)_N &= -kTn\left(\frac{\partial V}{\partial \mu}\right)_N.
\end{aligned}$$

Given: $(\partial V/\partial \mu) = -\overline{N}K_T$

$$\Rightarrow +(kTn)\overline{N}K_T = \left(K_T/K_T^0\right)\overline{N}.$$

Hence

$$\frac{\langle (N - \overline{N})^2\rangle}{\overline{N}} = K_T/K_T^0.$$

Evaluation of $\left(\frac{\partial n}{\partial \mu}\right)_V$. Use calculus identity:

$$\left(\frac{\partial \omega}{\partial y}\right)_x = \left(\frac{\partial \omega}{\partial y}\right)_z + \left(\frac{\partial \omega}{\partial z}\right)_y\left(\frac{\partial z}{\partial y}\right)_x.$$

$$\Rightarrow \left(\frac{\partial n}{\partial \mu}\right)_V = \left(\frac{\partial n}{\partial \mu}\right)_N + \left(\frac{\partial n}{\partial N}\right)_\mu \left(\frac{\partial N}{\partial \mu}\right)_V .$$

Last term is zero as $n = \overline{N}/V$ doesn't depend on N.

Therefore

$$\left(\frac{\partial n}{\partial \mu}\right)_V = \left(\frac{\partial n}{\partial \mu}\right)_N ,$$

hence

$$\left(\frac{\partial n}{\partial \mu}\right)_{T,V} = \frac{-\overline{N}}{V^2} \cdot \left.\frac{\partial V}{\partial \mu}\right)_N ,$$

as required.

5. At T_c, P_c

$$\frac{\partial P}{\partial V} = 0,$$

therefore:

$$K_T \sim 1/\frac{\partial P}{\partial V} \to \infty.$$

Thus at T_c, P_c,

$$K_T \to \infty, \text{ and so } \langle (\Delta N)^2 \rangle \to \infty.$$

Consider relationship between density fluctuations and density-density correlation $G(\mathbf{r} - \mathbf{r}')$. Defining $N = \int_V d\mathbf{r} n(r)$, we have

$$\langle (N - \overline{N})^2 \rangle = \langle \int d\mathbf{r}\{n(\mathbf{r}) - \langle n(\mathbf{r}) \rangle\}| \int d\mathbf{r}'\{n(\mathbf{r}') - \langle n(\mathbf{r}') \rangle\}\rangle,$$

and from the definition of G

$$\begin{aligned}
\langle (N - \overline{N})^2 \rangle &= \int d\mathbf{r} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \\
&= \int d\mathbf{r} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \\
&= \int d\mathbf{r} \int d\mathbf{r}'' G(\mathbf{r}'') \qquad \text{from spatial homogeneity} \\
&= V \int d\mathbf{r}'' G(\mathbf{r}'').
\end{aligned}$$

Combining with result of previous question:

$$K_T / K_T^0 = n^{-1} \int d\mathbf{r} G(\mathbf{r}).$$

L Solutions to Exercises C

1.

$$X(n+1) = f[X(n)].$$

Condition for a fixed point is

$$X(n+1) = X(n) = a$$

therefore substituting above for $X(n+1)$, and $X(n)$ gives

$$a = f(a),$$

as required.

(a)

$$X(n+1) = 2X(n) - 3$$

therefore

$$f[X(n)] = 2X(n) - 3$$

Hence $a = f(a) \Rightarrow a = 2a - 3$ and $a = 3$.

(b)

$$X(n+1) = rX(n) + b$$
$$f = rX(n) + b$$
$$f(a) = ra + b$$
$$a = ra + b \text{ for a fixed point}$$
$$a(1-r) = b$$
$$a = \frac{b}{1-r}.$$

For $r = 1,\ a = \infty$.

2. Given:

$$X(n+1) = [X(n) + 4]X(n) + 2$$
$$f(X(n)) = [X(n) + 4]X(n) + 2$$
$$f(a) = (a+4)a + 2$$
$$\text{Now, } a = f(a) \text{ for a fixed point, so}$$
$$a = (a+4)a + 2 \Rightarrow a = -1, -2.$$

Hence

$$\begin{aligned} X(0) &= -1.01 & \lim_{n\to\infty} X(n) &= -2; \\ X(0) &= -0.99 & \lim_{n\to\infty} X(n) &= \infty; \\ X(0) &= -2.4 & \lim_{n\to\infty} X(n) &= -2. \end{aligned}$$

See Section 1.6.3 for the rest of this solution.

3. Logistic equation is

$$X(n+1) = (1+r)X(n) - bX^2(n),$$

thus:

$$a = (1+r)a - ba^2 \quad \text{and so} \quad 0 = a(r - ba).$$

But

$$b = r/L,$$

hence

$$0 = a(r - r\frac{a}{L}),$$

and so

$$0 = a(1 - \frac{a}{L})r.$$

As $r \neq 0, a = 0$ or L.

i.e. If population zero, it remains zero. Whereas if it grows, L is the limit.

4. Consider spins on a lattice. For $T > T_c$, there is disorder on scales $> \xi$ where ξ is finite. Coarse-graining effectively reduces ξ and the system flows under RGT to complete disorder, which is the high-temperature fixed point. Accordingly the high-temperature fixed point is *attractive.*

   For $T < T_c$, the reverse argument holds and the low-temperature fixed point - which corresponds to perfect order - is also attractive.

   In parameter space there must be a 'watershed', which separates the trajectories to the low-T and high-T fixed points. This is the critical surface and contains the *mixed* or *critical* fixed point.

   For one-dimensional Ising, the situation is skewed, as the $T = 0$ fixed point is also the critical fixed point. For $T > 0$, RGT flow is to the fixed point at $T \to \infty$.

5. Partition sum

$$\mathcal{Z} = 1 + 2e^{-\beta\epsilon_1} + 2e^{-\beta\epsilon_2} + e^{\beta\epsilon_3}.$$

   Mean energy

$$\langle E \rangle = \frac{\left(2\epsilon_1 e^{-\beta\epsilon_1} + 2\epsilon_2 e^{-\beta\epsilon_2} + \epsilon_3 e^{-\beta\epsilon_3}\right)}{\mathcal{Z}}.$$

   Mean-square energy

$$\langle E^2 \rangle = \frac{\left(2\epsilon_1^2 e^{-\beta\epsilon_1} + 2\epsilon_2^2 e^{\beta\epsilon_2} + \epsilon_3^2 e^{-\beta\epsilon_3}\right)}{\mathcal{Z}}.$$

6. As the particles are on a lattice they may be treated as distinguishable. Hence:

$$\mathcal{Z}_{dis} = (\mathcal{Z}_1)^N,$$

   where $\mathcal{Z}_1$ is the partition function for any one spin. Now

$$\begin{aligned} \mathcal{Z}_1 &= \sum_j e^{-\epsilon_j/kT} \\ &= e^{-\mu B/kT} + e^{+\mu B/kT} \\ &= 2\cosh(\mu B/kT). \end{aligned}$$

   Thus

$$\mathcal{Z}_{dis} = [2\cosh(\mu B/kT)]^N.$$

   Helmholtz free energy F

$$F = -kT \ln \mathcal{Z}_{dis} = -NkT \ln[2\cosh(\mu b/kT)].$$

   Internal energy $U = \langle E \rangle$

$$\begin{aligned} U = \langle E \rangle &= kT^2 \frac{\partial}{\partial T} \ln \mathcal{Z}_{dis} \\ &= -N\mu B \tanh(\mu B/kT). \end{aligned}$$

   Entropy

$$\begin{aligned} S &= (\overline{E} - F)/kT \\ &= Nk\{\ln[2\cosh \mu B/kT - \mu B/kT \tanh(\mu B/kT)]\}. \end{aligned}$$

   Magnetic moment M

$$\begin{aligned} M = N\overline{\mu} &= N \sum_i \mu_i p_i \\ &= N \frac{\left[\mu e^{\mu B/kT} - \mu e^{-\mu B/kT}\right]}{e^{\mu B/kT} + e^{\mu B/kT}} = N_\mu \tanh \mu B/kT. \end{aligned}$$

   Specific heat C_B

$$C_B = \left(\frac{\partial \overline{E}}{\partial T}\right)_B = Nk\left(\mu B/kT\right)^2 / \cosh^2\left(\mu B/kT\right).$$

M Solutions to Exercises D

1. First part covered in Appendix C.

   Define:
$$G_n(r) = \langle S_n S_{n+r} \rangle.$$
   r is distance between sites, measured in units of lattice constant a.
$$G_n(r) = \mathcal{Z}_N^{-1} \sum_{\{s\}} S_n S_{n+r} e^{\sum_{i=1}^{N-1} K_i S_i S_{i+1}}; \quad K_i \equiv \frac{J_i}{kT}.$$
   Re-write
$$\mathcal{Z}_N G_n(r) = \sum_{\{s\}} S_n S_{n+r} e^{\sum K_i S_i S_{i+1}}.$$
   Consider nearest neighbour case: $r = 1$.
$$\begin{aligned}\mathcal{Z}_N G_n(1) &= \sum_{\{s\}} S_n S_{n+1}\, e^{\sum K_i S_i S_{i+1}} = \frac{\partial}{\partial K_n} \sum_{\{s\}} e^{\sum K_i S_i S_{i+1}} \\ &= \frac{\partial}{\partial K_N} Z_N\end{aligned}$$
   and inductively
$$Z_N G_n(r) = \frac{\partial}{\partial K_n} \frac{\partial}{\partial K_{n+1}} \cdots \frac{\partial}{\partial K_{n+r-1}} \mathcal{Z}_N.$$
   Hence:
$$2^N \Pi_{i=1}^{N-1} \cosh K_i G_n(1) = 2^N \Pi_{i=1}^{N-2} \cosh K_i \times \sinh K_n,$$
   therefore
$$G_n(1) = \tanh K_n,$$
$$G_n(r) = \Pi_{i=1}^{r} \tanh K_{n+i-1}.$$
   Uniform interaction $\Rightarrow G_n(r) = \tanh^r K$.

   Consider limit $r \to \infty$. $T > 0$, $\tanh K < 1$ and $G_n \to 0$ as $r \to \infty$. Hence for $T = 0$, $K \to \infty$ and $G_n \to 1$.

2. Generalize Hamiltonian:
$$H = -\sum_{i=1}^{N-1} J S_i S_{i+1} \qquad \to \qquad H = -\sum_{i=1}^{N-1} J S_i S_{i+1} - B \sum_{i=1}^{N} S_i.$$
   Generalize transfer function:
$$f(S_i, S_{i+1}) = e^{-\beta H(S_i, S_{i+1})} = e^{+\beta J S_i S_{i+1} + \beta B (S_i + S_{i+1})/2}.$$
   Let $\beta J \equiv K$, and $\beta B \equiv b$.

   Transfer matrix
$$\mathbf{T} \equiv \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} = \begin{pmatrix} e^{K+b} & e^{-K} \\ e^{-K} & e^{K-b} \end{pmatrix}$$
   where matrix elements are generated from:
$$T_{\pm\pm} \equiv f(S_i = \pm 1, S_{i+1} = \pm 1).$$
   Eigenvalues:
$$\lambda_{\pm} = e^K \cosh b \pm \left(e^{2K} \sinh^2 b + e^{-2K} \right)^{1/2}.$$
$$\mathcal{Z}_N = \lambda_+^N + \lambda_-^N$$

and as $N \to \infty$, only larger eigenvalue λ_+ is relevant.

Free energy per lattice site: $f = -\frac{1}{\beta}\ln\lambda_+$, and so:

$$f = -K - \beta^{-1}\ln\left[\cosh b \pm \left(e^{2K}\sinh^2 b + e^{-2K}\right)^{1/2}\right].$$

$$m = \langle s\rangle = -\beta\frac{\partial f}{\partial b}$$

$$= \frac{\sinh b \pm \sinh b\cosh b\left(e^{2K}\sinh^2 b + e^{-2K}\right)^{-1/2}}{\left[\cosh b \pm \left(e^{2K}\sinh^2 b + e^{-2K}\right)^{1/2}\right]}.$$

As $b \to 0,\ m \to 0$ for all T. Except $T = 0$ when one can have a finite permanent magnetization.

3.

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)} \qquad (1); \qquad y' = \frac{y(x+y)}{(1+xy)} \qquad (2).$$

$$(x*, y*) = (0,1) \qquad \text{Eqn}(1) \ \to 0 = \frac{0\times 1}{1} = 0 \qquad \text{eqn}(2) \to \qquad 1 = \frac{1(0+1)}{1+0} = 1$$

$$(x*, y*) = (0,0) \qquad \text{Eqn}(1) \ \to 0 = \frac{0\times 1}{1} = 0 \qquad \text{eqn}(2) \to \qquad 0 = \frac{0\times 0}{1} = 0$$

$$x* = 1,\ 0 \le y* \le 1 \qquad \text{Eqn}(1) \qquad \to \qquad 1 = \frac{1(1+y)^2}{(1+y)^2} = 1 \qquad \forall y \qquad \text{eqn}(2) \Rightarrow y' = y.$$

$$x = 1 \leftrightarrow T = \infty;\ y = 0,\ B = \infty,\ y = 1,\ B = 0,$$

therefore ferromagnetic fixed point is $(0,1)$.

4. Each spin on a square lattice interacts with the 4 nearest neighbours on a square round it. Denote alternate spins on lattice by r_i, remainder by t_i :

$$\{S_i\} = \{r_i\} + \{t_i\}.$$

Partition sum can be expressed in terms like

$$\sum_{\{r_i\}} \dots e^{Kr_i(t_1+t_2+t_3+t_4)} \dots$$

and if we do the sum over $r_i = \pm 1$

$$\mathcal{Z} = \sum_{\{t_i\}} \dots \left[e^{K(t_1+t_2+t_3+t4)} + e^{-K(t_1+t_2+t_3+t_4}\right] \dots$$

where $t_1 - t_4$ are the nearest neighbour spins to each r_i.

i.e. partition function is the product of many similar terms to that.

Now re-label the remaining spins S_i. Our renormalization condition is:

$$\left[e^{K(S_1+S_2+S_3+S_4)} \quad + \quad e^{-K(S_1+S_2+S_3+S_4)}\right]$$
$$= \quad fe^{\left\{\frac{K_1}{2}(S_1S_2+S_2S_3+S_3S_4+S_4S_1)+K_2(S_1S_3+S_2S_4)+K_3S_1S_2S_3S_4\right\}}.$$

This must hold for all possible values of S_1, S_2, S_3 and S_4

e.g. *all* $S_i = +1$ or *all* $= -1$ gives:

$$e^{4K} + e^{-4K} = fe^{2K_1+2K_2+K_3}$$

and other possible combinations:

$$2 = fe^{-2K_1+2K_2+K_3};$$
$$e^{2K} + e^{-2K} = fe^{-K_3};$$
$$2 = fe^{-2K_2+K_3}.$$

Note S_i are spins on the *new* lattice. Hence nearest-neighbour terms like S_1S_2 will appear in *one* other square bracket. This gives a factor of 2 so that total contribution from S_1 and S_2 is

$$e^{K_1S_1S_2}.$$

In all, for N spins on original lattice

$$\mathcal{Z} = f(K)^{N/2} \sum_{states} \exp\{K_1 \sum_{\langle ij \rangle} S_iS_j\} + K_2 \sum_{\langle\langle i,j \rangle\rangle} S_iS_j + K_3 \sum_{ij_{kl}} S_iS_jS_kS_l,$$

where:

$$K_1 = \frac{1}{4}\ln\cosh(4K); \qquad K_2 = \frac{1}{8}\ln\cosh(4K)$$
$$K_3 = \frac{1}{8}\ln\cosh(4K) - \frac{1}{2}\ln\cosh(2K).$$

N Solutions to Exercises E

1. For this you need to draw the graphs as discussed in Section 3.1.5.

2. Take $x = 0$ on the central plane, with $n_+(0) = n_-(0) = n_0$ (say). Generalise (3.22), (3.28): first $n(x) = n_0e^{-e\phi/kT}$ and second (for positive and negative charges) we have

$$en(x) = e(n_+ - n_-) = en_+(0)e^{-e\phi/kT} - en_-(0)e^{e\phi/kT}$$
$$= -2en_0\sinh\{e\phi/kT\}.$$

Take Poisson's equation as

$$\frac{d^2\phi}{dx^2} = 4\pi en(x) = 8\pi en_0\sinh\{e\phi/kT\}$$
$$\simeq \frac{8\pi e^2n\phi}{kT} \quad \text{as} \quad e\phi \ll kT.$$

and from (3.31)

$$= \frac{2\phi}{l_D^2}; \quad \text{where} \quad l_D = \left[\frac{4\pi e^2n_0}{kT}\right]^{1/2}.$$

The solution of this differential equation can be written as

$$\phi = A\sinh(\sqrt{2}x/l_D) + B\cosh(\sqrt{2}x/l_D).$$

With boundary conditions $\phi = 0$ at $x = 0$, it is easily seen that we must have $B = 0$. Also,

$$\phi = \pm 1/2V \quad \text{at} \quad x = \pm a,$$

so

$$A = \frac{V}{2}\frac{1}{\sinh(\sqrt{2}a/l_D)},$$

and

$$\phi = \frac{V}{2}\cdot\frac{\sinh(\sqrt{2}x/l_D)}{\sinh(\sqrt{2}a/l_D)}.$$

Substitute back into the expression for the space charge:

$$n(x) \simeq \frac{-2e^2n_0\phi}{kT} = \frac{-e^2n_0}{kT}V\frac{\sinh(\sqrt{2}x/l_D)}{\sinh(\sqrt{2}a/l_D)}.$$

3. Equilibrium magnetization corresponds to minimum free energy. From Section 3.3.2.

$$\begin{aligned}\frac{dF}{dM} &= 2A_2M + 4A_4M^3 = 0 \\ &= 2A_{20}(T - T_c)M + 4A_4M^3,\end{aligned}$$

hence:

$$M = 0 \text{ or } M^2 \sim (T - T_c)$$

thus

$$M \sim (T - T_c)^{1/2} \sim \theta_c^{1/2},$$

and so

$$\beta = 1/2.$$

To obtain γ and δ add a magnetic term due to external field B :

$$F = F_0 + A_{20}(T - T_c)M^2 + A_4M^4 - BM$$

therefore

$$\frac{dF}{dM} = -B + 2A_{20}(T - T_c)M + 4A_4M^3 = 0.$$

For critical isotherm, $T = T_c$ and so $B \sim M^3$, thus: $\delta = 3$.

Now

$$\chi = \left.\frac{\partial M}{\partial B}\right)_T$$

to differentiate both sides of equation for equilibrium magnetization w.r.t B :

$$1 = 2A_{20}(T - T_c)\left.\frac{\partial M}{\partial B}\right)_T + 12A_4M^2\left.\frac{\partial M}{\partial B}\right)_T$$

and

$$\chi = \left(2A_2T_c\theta_c + 12A_4M^2\right)^{-1} \Rightarrow \gamma = 1$$

O Solutions to Exercises F

1. In Chapter Four we derived the result for the closed Ising chain as eqn (4.20) which may be written in the form:

$$\mathcal{Z}_N = 2^N \cosh^N K\left(1 + \tanh^N K\right).$$

In the case of the open chain, the only graph with an even number of vertices is the zero-order. Hence the above result becomes

$$Z_N = 2^N \cosh^{N-1} K.$$

Note that in the case of an open chain we have $P = N - 1$ as the number of pairs of lattice sites.

2. By definition:

$$\begin{aligned}\langle S_mS_n\rangle &= \mathcal{Z}_N^{-1}\sum_{\{S\}} S_mS_ne^{-\beta H} \\ &= \mathcal{Z}_N^{-1}\sum_{\{S\}} S_mS_n\prod_{\langle i,j\rangle} e^{KS_iS_j}.\end{aligned}$$

The argument proceeds as for the partition function, and using the identity given as eqn (4.4):

$$\langle S_mS_n\rangle = \mathcal{Z}_N^{-1}\cosh^P K\sum_{\{S\}}\Pi_{\langle i,j\rangle}S_mS_n(1 + vS_iS_j)$$

Figure 1: .

where $P \equiv N^0$ of nearest-neighbour pairs.

Now introduce lattice graphs in one-to-one correspondence with the terms in the expansion. As before, require *every* spin in a product is to be raised to an *even* power to have a non-zero contribution. *But* as have factor $S_m S_n$, for these two sites we require an *odd* power of the spin.

Hence we replace $g(r)$ in the expansion for the partition function by $f_{mn}(r)$, where $f_{mn}(r)$ is the same as $g(r)$ except at sites m and n.

Consider a linear Ising chain. $P = N - 1$.

For an open chain there are *no* closed graphs and hence the only contribution to the partition function would be the zero-order graph.

therefore

$$\mathcal{Z}_N = 2^N \cosh^{N-1} K.$$

(Compare to the result in Section 4.2 for the *closed* Ising ring.)

Because of the factor $S_n S_m$, the v^0 term doesn't contribute to the correlation. The only non-vanishing contribution is from the term shown in the figure.

Note

(1) all connected vertices except m and n are even

(2) vertices m and n are odd.

This term has $n - m$ vertices and is therefore of order $|n - m|$.

$$\begin{aligned} \langle S_m S_n \rangle &= \mathcal{Z}_N^{-1} \cosh^P K 2^N \sum_{r=1}^{P} f_{mn}(r) v^r \\ &= \left(2^N \cosh^{N-1} K\right)^{-1} \left(2^N \cosh^{N-1} K\right) v^{|n-m|} \\ \text{therefore} \quad \langle S_m S_n \rangle &= v^{|n-m|} = \tanh^{|n-m|} K. \end{aligned}$$

3. The Van der Waals equation takes the form

$$\left(P + \frac{a}{V^2}\right)(V - b) = NkT,$$

where a/V^2 represents the effect of mutual attraction between molecules and b is the 'excluded volume' due to the finite size of the molecules. The equation is based on a model where $\phi(r)$ is taken as corresponding to a 'hard sphere' potential for $r \leq d$, but is weakly attractive for $r > d$.

From eqns (4.63) and (4.58) we have

$$B_2 = -\frac{1}{2} I_2 = -\frac{1}{2} \int d\mathbf{r} \left[e^{-\phi(r)/kT} - 1\right],$$

and, on the basis of our assumptions about $\phi(r)$, we may make the simplification

$$\begin{aligned} e^{-\phi(r)/kT} &\simeq 0 \quad \text{for } r < d; \\ &\simeq 1 - \phi/kT, \quad \text{for } r > d. \end{aligned}$$

Then, dividing the range of integration into two parts, we obtain

$$B_2 = \frac{1}{2}\int_0^d 4\pi r^2 dr + \frac{1}{2}\int_d^\infty 4\pi r^2 \frac{\phi(r)}{kT} dr = B - A/kT,$$

where

$$B = \frac{2\pi d^3}{3} = 4v_0,$$

where $v_0 \equiv$ volume of a molecule, and

$$A = -2\pi \int_d^\infty r^2 \phi(r) dr.$$

Now, from (4.62) and these results, the pressure is given by

$$P = \frac{NkT}{V}\left[1 + \frac{N}{V}\left(B - \frac{A}{kT}\right)\right].$$

Let us define the Van der Waals constants as:

$$b = 4Nv_0,$$

which is the total excluded volume of N molecules, and

$$a = N^2 A,$$

which is the total effect of interactions between all possible pairs. Then the equation for the pressure becomes

$$P = \frac{NkT}{V} + \frac{NkT}{V}\cdot\frac{b}{V} - \frac{a}{V^2};$$

or:

$$P + \frac{a}{V^2} \simeq \frac{NkT}{V}\cdot\frac{1}{1 - b/V},$$

where the interpretation in terms of a first-order truncation of the binomial expansion is justified for small values of b/V. Then multiplying across, and cancelling as appropriate, yields

$$\left(P + \frac{a}{V^2}\right)(V - b) = NkT,$$

as required.

4. From (4.63) and (4.58),

$$B_2 = -\frac{1}{2}\int d^3 r f(r).$$

The hard sphere potential satisfies

$$\begin{aligned} \Phi(r) &= \infty \qquad r < a; \\ &= 0 \qquad r > a. \end{aligned}$$

Hence $f(r)$ satisfies:

$$\begin{aligned} f(r) &= -1 \qquad r < a; \\ &= 0 \qquad r > a. \end{aligned}$$

Thus the integral becomes:

$$\begin{aligned} B_2 &= \left(-\tfrac{1}{2} \times -1\right) 4\pi \int_0^a r^2 dr \\ &= \frac{2\pi a^3}{3}, \end{aligned}$$

as required.

For B_3, start by doing the integral with respect to r', thus:

$$H(r) = \int d^3r' f(r') f(|\mathbf{r} - \mathbf{r}'|)$$

The integrand is $-1 \times -1 = 1$ for r' lying within a sphere of radius a centred on the origin and also within a sphere of radius a centred on $r' = r$. Thus $H(r)$ is the volume of space occupied by the two overlapping spheres, and by symmetry consists of two spherical 'caps', on each side of a plane through $r' = r/2$.

Let ρ be a coordinate lying between $r/2$ and a, corresponding to one half of the volume of intersection. Then it is easily seen that

$$\begin{aligned} H(r) &= 2\int_{r/2}^{a} \pi(a^2 - \rho^2) d\rho \\ &= \pi\left(\frac{4a^3}{3} - a^2 r + \frac{r^3}{12}\right) . \end{aligned}$$

Finally,

$$B_3 = \frac{1}{3} 4\pi \int_0^a r^2 H(r) dr = \frac{5\pi^2 a^6}{18} = \frac{5}{8} B_2^2 ,$$

as required.

5. The relationship has been written to make the problem look more difficult. Start by inverting it:

$$\lambda^{3N(\frac{1}{n}-\frac{1}{2})} Z(\lambda T, \lambda^{-3/n} V) = Z(T, V) .$$

The right hand side is our starting point: from eqns (4.31), (4.32), (4.33) and (4.34) we have

$$Z(T,V) = \frac{1}{h^N N!}(2\pi m k T)^{\frac{3N}{2}} \int d^N \mathbf{q} \exp\left[-\frac{A}{kT}\sum_{ij}\frac{1}{|\mathbf{q}_i - \mathbf{q}_j|^n}\right] ,$$

where we have substituted the given potential. Look at the Boltzmann exponent first. We want $T \to \lambda T$, so we make the change of variables:

$$\mathbf{q}_i = \lambda^{1/n} \mathbf{x}_i ,$$

and the exponent becomes:

$$\frac{A}{kT}\sum_{ij} \frac{1}{(\lambda^{1/n}|\mathbf{x}_i - x_j|)^n} = -\frac{A}{k\lambda T}\sum_{ij}\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|^n} .$$

Now look at the other consequences of the change of variable:

$$\begin{aligned} d\mathbf{q}_i &= \lambda^{3/n} d\mathbf{x}_i \\ V = \int d\mathbf{q}_i &= \lambda^{3/n} \int d\mathbf{x}_i = \lambda^{3/n} V' \\ V' &= \lambda^{-3/n} V \end{aligned}$$

Now, ignoring the constant prefactor $(h^N N!)^{-1}$, which will cancel across, we have:

$$\begin{aligned}
Z(T,V) &= (2\pi mkT)^{\frac{3N}{2}}\lambda^{\frac{3N}{n}}\int d^N\mathbf{x}\exp\left[\frac{-A}{k\lambda T}\sum_{ij}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|^n}\right] \\
&= (2\pi mk\,\lambda T)^{\frac{3N}{2}}\lambda^{3N(\frac{1}{n}-\frac{1}{2})}\int d^N\mathbf{x}\exp\left[\frac{-A}{k\lambda T}\sum_{ij}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|^n}\right] \\
&= \lambda^{3N(\frac{1}{n}-\frac{1}{2}}Z(\lambda T,\lambda^{-3/n}V)\,,
\end{aligned}$$

as required.

6.

$$\frac{\partial}{\partial x}e^{-\beta H} = -\int_0^\beta e^{-(\beta-y)H}\frac{\partial H}{\partial x}e^{-yH}dy.$$

Note that the variable y is necessarily $\mathcal{O}(\beta)$, and expand both sides as power series in β.

$$\begin{aligned}
L.H.S. &= \frac{\partial}{\partial x}[1-\beta H+(\beta H)^2/2!+\ldots] \\
&= -\beta\frac{\partial H}{\partial x}+\frac{\beta^2}{2!}H\frac{\partial H}{\partial x}+\frac{\beta^2}{2!}\frac{\partial H}{\partial x}H+\ldots.
\end{aligned}$$

$$\begin{aligned}
R.H.S. &= -\int_0^\beta[1-(\beta-y)H+\frac{1}{2!}(\beta-y)^2H^2+\ldots] \\
\times &\ \frac{\partial H}{\partial x}[1-yH+\frac{y^2}{2!}H^2+\ldots]dy \\
&= -\int_0^\beta\frac{\partial H}{\partial x}dy+\int_0^\beta\left[(\beta-y)H\frac{dH}{dx}+\frac{\partial H}{\partial x}yH\right]dy+\ldots \\
&= -\int_0^\beta\frac{\partial H}{\partial x}dy+\int_0^\beta\beta H\frac{\partial H}{\partial x}dy-\int_0^\beta yH\frac{dH}{dx}dy+\int_0^\beta\frac{\partial H}{\partial x}yHdy+\ldots \\
&= -\beta\frac{\partial H}{\partial x}+\{\beta^2H\frac{\partial H}{\partial x}-\frac{\beta^2}{2}H\frac{\partial H}{\partial x}\}+\frac{\beta^2}{2}\frac{\partial H}{\partial x}H+\ldots \\
&= -\beta\frac{\partial H}{\partial x}+\frac{\beta^2}{2}H\frac{\partial H}{\partial x}+\frac{\beta^2}{2}\frac{\partial H}{\partial x}H+\ldots \\
&= L.H.S.
\end{aligned}$$

7. For the canonical ensemble in the energy representation, we have:

$$C_V=\left.\frac{\partial\overline{E}}{\partial T}\right)_V, \qquad \overline{E}=kT^2\frac{\partial}{\partial T}\ln\mathcal{Z}$$

where

$$\begin{aligned}
\mathcal{Z} &= Tre^{-\beta H} \\
&= Tr\left[1-\beta H+\frac{\beta^2}{2!}H^2+0(\beta^3)\right] \\
&= \left[Tr1-\beta TrH+\frac{\beta^2}{2}TrH^2+0(\beta^3)\right] \\
&= Tr1\left[1-\beta\frac{TrH}{Tr1}+\frac{\Gamma^2}{2}\frac{TrH^2}{Tr1}+0(\beta^3)\right].
\end{aligned}$$

Then

$$\begin{aligned}
\overline{E} &= kT^2\frac{\partial}{\partial T}\ln\{Tr1\left[1-\frac{\beta TrH}{Tr1}+\frac{\beta^2}{2}\frac{TrH^2}{Tr1}+0(\beta^3)\right]\} \\
&= kT^2\frac{\partial}{\partial T}\{\ln Tr1+\ln\left[1-\frac{\beta TrH}{Tr1}+\frac{\beta^2}{2}\frac{TrH^2}{Tr1}+0(\beta^3)\right]\} \\
&= kT^2\frac{\partial}{\partial T}\ln[1-P+0(\beta^3)] \\
\text{where } P &= \frac{\beta TrH}{Tr1}+\frac{\beta^2}{2}\frac{TrH^2}{Tr1}.
\end{aligned}$$

Expand $\ln\left[1-P+0(\beta^3)\right]$ to $0(\beta^3)$, differentiate (the βs) w.r.t T and required result follows.

8.

$$\left(P+\frac{a}{V^2}\right)(V-b)=NkT$$

Critical point $\leftrightarrow$ inflection of isotherms therefore,

$$\left.\frac{\partial P}{\partial V}\right)_{T=T_c}=\left.\frac{\partial^2 P}{\partial V^2}\right)_{T=T_c}=0$$

gives conditions.

Re-arrange VdW equation:

$$P=\frac{NkT}{V-b}-\frac{a}{V^2}, \qquad (*)$$

$$\left.\frac{\partial P}{\partial V}\right)_T=\frac{-NkT}{(V-b)^2}+\frac{2a}{V^3}=0 \text{ for } T=T_c, V=V_c.$$

Thus:

$$\frac{2a}{V_c^3}=\frac{NkT_c}{(V_c-b)^2};$$

$$\frac{2a(V_c-b)^2}{V_c^3}=NkT_c; \qquad (**)$$

$$\left.\frac{\partial^2 P}{\partial V^2}\right)_T=\frac{2NkT}{(V-b)^3}-\frac{6a}{V4}=0 \text{ for } T=T_c, V=V_c;$$

and

$$\frac{3a}{V_c^4}=\frac{NkT_c}{(V_c-b)^3},$$

hence

$$\frac{3a(V_c-b)^3}{V_c^4}=NkT_c. \qquad (***)$$

Equate LHS of (**) and (***):

$$2a\frac{(V_c-b)^2}{V_c^3}=3a\frac{(V_c-b)^3}{V_c^4}$$

$$\Rightarrow V_c=3b.$$

Substituting for V_c into (**) $\Rightarrow NkT_c=8a/27b$ and both V_c and T_c in (*) $\Rightarrow P_c=a/27b^2$.

Set $\tilde{p}=P/P_c$, $\tilde{v}=V/V_c$ and $\tilde{t}=T/T_c$.

Hence put $P=P_c\tilde{p}$, $V=V_c\tilde{v}$ and $T=T_c\tilde{t}$ in equation (*) and substitute for P_c, V_c and T_c, to get:

$$\left(\frac{a}{27b^2}\tilde{p}+\frac{a}{\tilde{v}^2 9b^2}\right)(3b\tilde{v}-b)=Nk\tilde{t}\frac{8a}{27b}\frac{1}{Nk}$$

$$\Rightarrow \frac{a}{27b^2}\left(\tilde{p}+\frac{3}{\tilde{v}^2}\right)\times 3b\left(\tilde{v}-\frac{1}{3}\right)=\frac{8a\tilde{t}}{27b},$$

and so

$$\left(\tilde{p}+\frac{3}{\tilde{v}^2}\right)(\tilde{v}-1/3)=\frac{8\tilde{t}}{3}.$$

9. From the previous question:

$$\left(\tilde{p}+\frac{3}{\tilde{v}^2}\right)(3\tilde{v}-1)=8\tilde{t}.$$

Set;

$$\begin{array}{rclcc} p &=& \tilde{p}-1=(P-P_c)/P_c & \text{hence} & \tilde{p}=p+1 \\ v &=& \tilde{v}-1=(V-V_c)/V_c & \text{hence} & \tilde{v}=V+1 \\ \Theta_c &=& \tilde{t}-1=(T-T_c)/T_c & \text{hence} & \tilde{t}=\theta_c+1 \end{array}$$

$$\Rightarrow \left[p+1+\frac{3}{(1+v)^2}\right][3(1+v)-1]=8(1+\theta_c).$$

Multiply through by $(1+v)^2$, multiply out and re-arrange:

$$2p\left(1+\frac{7v}{2}+4v^2+\frac{3v^3}{2}\right)=-3v^3+8\theta_c(1+2v+v^2)$$

or

$$p=-\frac{3}{2}v^3+\theta_c(4-6v+9v^2+\ldots)+\ldots,$$

hence

$$\gamma\left(-VK_T^{-1}\right)=+\left(\frac{\partial V}{\partial P}\right)_T=\frac{P_c}{V_c}\left(\frac{\partial p}{\partial v}\right)_T=\frac{P_c}{V_c}(-6\theta_c)$$

and so

$$K_T\sim\theta_c^{-1}\text{ therefore }\gamma=1.$$

δ. Critical isotherm $T=T_c,\theta_c=0$ therefore $p\sim v^3$ hence $\delta=3$.

Note These are mean-field values and Van der Waals is nowadays re-interpreted as a mean-field theory.

P Solutions to Exercises G

1. Given

$$\chi_T(C_B-C_M)=T\left(\frac{\partial M}{\partial T}\right)_B^2$$

therefore

$$C_B=\frac{T\left(\frac{\partial M}{\partial T}\right)_B^2}{\chi_T}+C_M$$

Now C_M must be positive i.e. $C_M\geq 0$

therefore

$$C_B\geq\frac{T\left(\frac{\partial M}{\partial T}\right)_B^2}{\chi_T}\quad(*).$$

Definitions of critical exponents:

$$C_B\sim|\theta_c|^{-\alpha},\ \chi_T\sim|\theta_c|^{-\gamma},$$

$$M\sim(-\theta_c)^{\beta}\qquad\Rightarrow\qquad\left.\frac{\partial M}{\partial T}\right)_B\sim(-\theta_c)^{\beta-1}.$$

Now

$$(-\theta_c)^2 = |\theta_c|^2$$

Thus inequality (*) becomes

$$|\theta_c|^{-\alpha} \geq |\theta_c|^{2(\beta-1)+\gamma}.$$

Hence

$$-\alpha \leq 2(\beta - 1) + \gamma,$$

thus

$$\alpha + 2\beta - 2 + \gamma \geq 0,$$

and so

$$\alpha + 2\beta + \gamma \geq 2.$$

2. The Widom scaling relation is:

$$G(\lambda^\gamma \theta_c, \lambda^p B) = \lambda G(\theta_c, B).$$

Differentiate l times on both sides and re-arrange:

$$G^l(\theta_c, B) = \lambda^{lp-1} G^{(l)}(1)^\gamma(\theta_c, \lambda^p B.$$

Hence it follows that:

$$\begin{aligned}
\frac{G^{(l)}(\theta_c B)}{G^{(l-1)}(\theta_c, B)} &= \frac{\lambda^{lp-1} G^{(l)}\left(\lambda^\gamma \theta_c, \lambda^p B\right)}{\lambda^{(l-1)p-1} G^{(l-1)}\left(\lambda^\gamma \theta_c, \lambda^p B\right)} \\
&= \frac{\lambda^p G^{(l)}\left(\lambda^\gamma \theta_c, \lambda^p B\right)}{G^{(l-1)}\left(\lambda^\gamma \theta_c, \lambda^p B\right)}.
\end{aligned}$$

Now $L.H.S. =\sim \theta_c^{-\Delta_l}$ (from definition).
Also choose $\lambda = \theta_c^{-1/r}$, thus:

$$\begin{aligned}
\theta_c^{-\Delta_l} &= \left(\theta_c^{-1/r}\right)^p \frac{G^l\left(1, \lambda^p B\right)}{G^{(l-1)}\left(1, \lambda^p B\right)} \\
&= \theta_c^{-p/r} \\
\text{therefore } \Delta_l &= p/r, \text{ independent of } l.
\end{aligned}$$

3. The mean magnetization $M = \langle S \rangle_0$ and from mean field theory:

$$\langle S \rangle_0 = \tanh(\beta B + 2zJ\beta\langle S \rangle_0).$$

Hence immediately we can write:

$$M = \tanh(\beta zJM + b) \text{ where } b \equiv \beta B.$$

Now mean field theory gives

$$z\beta_c J = 1 \text{ or } zJ = 1/\beta_c.$$

Hence

$$M = \tanh\left[\frac{\beta M}{\beta_c} + b\right] = \tanh\left[M\frac{T_c}{T} + b\right] = \tanh\left[M/(1+\theta_c) + b\right].$$

Set $B = 0$ and expand for $T \sim T_c$, in which case θ_c is small:

$$M = \frac{M}{1+\theta_c} - \frac{1}{3}\frac{M^3}{(1+\theta_c)^3},$$

thus

$$M\left(1 - \frac{1}{1+\theta_c}\right) = -\frac{1}{3}\frac{M^3}{(1+\theta_c)^3},$$

hence

$$M = 0$$

or

$$M^2 = -3\theta_c \frac{(1+\theta_c)^3}{(1+\theta_c)} = -3\theta_c(1+\theta_c)^2.$$

Taking the nontrivial case,

$$M \sim |-3\theta_c|^{1/2},$$

and by comparison with the equation which defines the critical exponent:

$$\beta = 1/2.$$

4. For $B = 0$, we have $H = -J\sum_{\langle i,j\rangle} S_i S_j$ where $\sum_{\langle i,j\rangle}$ is sum over nearest neighbours. The mean energy of the system is given by

$$\overline{E} = \langle H\rangle = -J\sum_{\langle i,j\rangle}\langle S_i S_j\rangle.$$

In lowest-order mean field approximation, spins are independent and

$$\langle S_i S_j\rangle = \langle S_i\rangle\langle S_j\rangle.$$

Hence

$$\overline{E} = -J\sum_{\langle i,j\rangle}\langle S_i\rangle\langle S_j\rangle = -Jz\frac{N}{2}m^2$$

where $M = \langle S\rangle \equiv$ order parameter. From the thermodynamic definition of the heat capacity, we have

$$C_B = \left.\frac{\partial\overline{E}}{\partial T}\right)_B = -2Jz\frac{N}{2}M\frac{dM}{dT} = -JzNM\frac{dM}{dT}.$$

For

$T > T_c$: $M = 0$ therefore $C_B = 0$;

$T \le T_c$: $M = (-3\theta_c)^{1/2}$.

Hence

$$\frac{\partial M}{\partial T} = \frac{1}{2}(-3\theta_c)^{-1/2} \times -\frac{d\theta_c}{dT} = \frac{-3}{2}M^{-1}\frac{d\theta_c}{dt} = \frac{-3}{2}M^{-1}T_c^{-1},$$

and so

$$\left.\frac{\partial\overline{E}}{\partial T}\right)_B = \frac{3}{2}JzNMM^{-1}T_c^{-1} = \frac{3}{2}\frac{JzN}{c}$$

$$= \frac{3}{2}Nk \quad \text{as} \quad Jz = kT_c.$$

Hence C_B is discontinuous at $T = T_c$ and $\alpha = 0$.

5. From the definition of the susceptibility, we have

$$\chi_T = \frac{\partial M}{\partial B} = \beta\frac{\partial M}{\partial b},$$

and also

$$M = \tanh\left(\frac{M}{1+\theta_c} + b\right) \simeq \frac{M}{1+\theta_c} + b \quad \text{for} \quad T > T_c.$$

Now

$$M - \frac{M}{1+\theta_c} = b,$$

to this order of approximation and, re-arranging, we have:

$$M = \left(\frac{1+\theta_c}{\theta_c}\right) b.$$

Hence

$$\chi_T \sim \frac{\partial M}{\partial b} \sim \frac{1}{\theta_c} \qquad \text{as} \qquad \theta_c \to 0$$

and so

$$\chi_T \sim \theta_c^{-1}, \qquad \gamma = -1.$$

Consider the effect of an externally imposed field at $T = T_c$, where $\theta_c = 0$, and so $1 + \theta_c = 1$. Use the identity:

$$\begin{aligned} M &= \tanh(M+b) = (\tanh M + \tanh b)(1 + \tanh M \tanh b) \\ &\simeq \left(M - \frac{M^3}{3} + b - \frac{b^3}{3}\right)(1 + \tanh M \tanh b) \end{aligned}$$

Cancel the factor of M on both sides and rearrange, to obtain:

$$b \sim \frac{M^3}{3} + \frac{b^3}{3} - \left(M - \frac{M^3}{3} + b - \frac{b^3}{3}\right)\left(Mb - \frac{Mb^3}{3} - \frac{Mb^3}{3} + \ldots\right)$$

therefore $b \sim M^3/b$ for small b, M : hence $\delta = 3$.

If we set $b \sim M^3$ on the right hand side, we can verify all terms of order higher than $\mathcal{O}\left(M^3\right)$ are neglected.

6. From eqn (7.72), the maximum free energy is:

$$F = F_0 - \frac{1}{2} NzJ\langle S\rangle_0^2 + B' N \langle S\rangle_0$$

where

$$B' = B_E - B;\ F_0 = \frac{-N}{\beta}\ln\left[2\cosh\left(\beta B_E\right)\right];\ \langle S\rangle_0 = \tanh\left(\beta B_E\right).$$

Substituting as appropriate:

$$F = \frac{-N}{\beta}\ln\left[2\cosh\left(\beta B_E\right)\right] - \frac{1}{2}NzJ\tanh^2\left(\beta B_E\right) + N\left(B_E - B\right)\tanh\left(\beta B_E\right).$$

The variational procedure gives:

$$\left(B_E - B\right) = zJ\langle S\rangle_0 = zJ\tanh\left(\beta B_E\right).$$

Rewrite this as

$$\tanh\left(\beta B_E\right) = \frac{B_E - B}{zJ}$$

and substitute into the expression for F :

$$\begin{aligned} F &= \frac{-N}{\beta}\ln\left[2\cosh\left(\beta B_E\right)\right] - \frac{1}{2}NzJ\frac{\left(B_E - B\right)^2}{4z^2J^2} + N\left(B_E - B\right)\frac{\left(B_E - B\right)}{zJ} \\ &= \frac{-N}{\beta}\ln\left[2\cosh\left(\beta B_E\right)\right] + \frac{N}{2zJ}\left(B_E - B\right)^2, \end{aligned}$$

as required.

7. Here we repeat the previous calculation with the generalization that the external magnetic field is no longer a constant but varies from one lattice site to the next and is denoted by B_i, for the ith lattice site.

It follows that the effective field $B_E \to B_E^{(i)}$ and the molecular field $B' \to B'_i$ also. Some of the key relationships also generalize straightforwardly, thus:

$$F_0 = -\beta^{-1} \sum_i \ln \left[2 \cosh \left(\beta B_E^{(i)} \right) \right], \tag{1}$$

and

$$\langle S_i \rangle_0 = \tanh \left(\beta B_E^{(i)} \right). \tag{2}$$

Then, from (1) and (2), we may easily obtain:

$$\frac{\partial F_0}{\partial B_E^{(i)}} = \sum_i \langle S_i \rangle_0. \tag{3}$$

Next we take eqn (7.68) with the equality and generalize to the nonuniform case, thus:

$$F = F_0 - \sum_{i,j} J_{ij} \langle S_i S_j \rangle_0 + \sum_i B'_i \langle S_i \rangle_0, \tag{4}$$

and, with an obvious generalization to the inhomogeneous case,

$$F = F_0 - \sum_{i,j} J_{ij} \langle S_i S_j \rangle_0 + \sum_i \left(B_E^{(i)} - B_i \right) \langle S_i \rangle_0. \tag{5}$$

As before, on the unperturbed model, we treat the spins as independent. Reminding ourselves of the properties of the double sum, we may write:

$$\sum_{i,j} J_{ij} \langle S_i S_j \rangle = \frac{J}{2} \sum_i \sum_{\langle j \rangle} \langle S_i \rangle_0 \langle S_j \rangle_0, \tag{6}$$

where $\langle j \rangle$ denotes 'sum over the nearest neighbours of each i'.

Now we carry out the variation, differentiating F as given by (5) and (6), with respect to $B_E^{(i)}$, thus:

$$\begin{aligned} \frac{\partial F}{\partial B_E^{(i)}} &= \frac{\partial F_0}{\partial B_E^{(i)}} - \frac{J}{2} \sum_i \sum_{\langle j \rangle} \frac{\partial \langle S_i \rangle_0}{\partial B_E^{(i)}} \langle S_j \rangle_0 + \frac{J}{2} \sum_i \sum_{\langle j \rangle} \langle S_i \rangle_0 \frac{\partial \langle S_j \rangle_0}{\partial B_E^{(i)}} \\ &+ \left(B_E^{(i)} - B_i \right) \frac{\partial \langle S_i \rangle_0}{\partial B_E^{(i)}} + \sum_i \langle S_i \rangle_0. \end{aligned} \tag{7}$$

Two points should now be noted:

(a) From (3), we see that the first term on the right hand side cancels the last term, just as in the homogeneous case.

(b) The second term involving the double sum vanishes because $\partial \langle S_j \rangle_0 / \partial B_E^{(i)} = 0$ for $j \neq i$, and j is **never** equal to i.

Then, setting $\partial F / \partial B_E^{(i)} = 0$, and equating coefficients of $\partial \langle S_i \rangle_0 / \partial B_E^{(i)}$, we obtain the condition for an extremum as:

$$B_E^{(i)} - B_i = J \sum_{\langle j \rangle} \langle S_j \rangle_0. \tag{8}$$

We may further write this condition in the useful form:

$$B_E^{(i)} - B_i = J \sum_{\langle j \rangle} \tanh \left(\beta B_E^{(j)} \right), \tag{9}$$

where we have substituted for $\langle S_i \rangle_0$ from (2).

We may conclude by writing down the optimal form of the free energy. From (5) and (6), along with (1) and (2), this is:

$$\begin{aligned} F &= -\frac{1}{\beta}\sum_i \ln\left[2\cosh\left(\beta B_E^{(i)}\right)\right] - \frac{J}{2}\sum_i\sum_{\langle j\rangle}\tanh\left(\beta B_E^{(i)}\right)\tanh\left(\beta B_E^{(j)}\right) \\ &+ \sum_i \left(B_E^{(i)} - B_i\right)\tanh\left(\beta B_E^{(i)}\right). \end{aligned} \qquad (10)$$

Then substituting from (9) into the middle term we have

$$F = -\frac{1}{\beta}\sum_i \ln\left[2\cosh\left(\beta B_E^{(i)}\right)\right] + \frac{1}{2}\sum_i \left(B_E^{(i)} - B_i\right)\tanh\left(\beta B_E^{(i)}\right). \qquad (11)$$

Note that this result is **not** a simple generalization of the result of the previous problem.

8. Mean-field approximation:

$$H = -(B + zJ\sigma)\sum_{i=1}^{N} S_i$$

where

$$\sigma = \langle S_i \rangle .$$

The partition function factorizes into a product of N single-spin partition functions, each given by:

$$Z_1 = \sum_{S=-t}^{t} e^{xS} = \frac{\sinh[(t+\frac{1}{2})x]}{\sinh[x/2]},$$

where

$$x = \beta(B + zJ\sigma).$$

Obtain a closed equation for σ by working out

$$\sigma = \langle S \rangle = \frac{1}{Z_1}\sum_{S=-t}^{t} S e^{xS} = \frac{d\ln Z_1}{dx}.$$

At $B = 0$,

$$\sigma = (t+\frac{1}{2})\coth\left[\left(t+\frac{1}{2}\right)\beta zJ\sigma\right] - \frac{1}{2}\coth\left(\frac{1}{2}\beta zJ\sigma\right).$$

By considering the form of the solution, conclude that there are non-zero solutions for σ where the slope of this function at $\sigma = 0$ is greater than unity:

$$\frac{t(t+1)}{3}\beta zJ > 1$$

or

$$T < T_c \equiv \frac{t(t+1)}{3}\frac{zJ}{k}.$$

For the Ising model, take $t = \frac{1}{2}$. Denoting Ising values by an overbar, define $\overline{S}_i = 2S_i$, such that $\overline{S}_i = \pm 1$ and

$$\overline{\sigma} = \langle \overline{S}_i \rangle = 2\sigma \, ,$$

along with $\overline{J} = J/4$, to give the correct interaction Hamiltonian. With these changes we find

$$\overline{\sigma} = \tanh(\beta z \overline{J}\, \overline{\sigma}) \, ,$$

and

$$\overline{T}_c = z\overline{J}/k \, .$$

9. Take:

$$\mathbf{B} = h\,\hat{\mathbf{z}}; \quad h > 0$$

$$\langle \mathbf{S}_j \rangle = \sigma\,\hat{\mathbf{z}} \, .$$

$$H = -qJ\sigma \sum_i \mathbf{S}_i.\,\hat{\mathbf{z}} - h \sum_i \mathbf{S}_i.\,\hat{\mathbf{z}} \, .$$

Here q is the coordination number of the lattice; normally we use z but that would cause confusion with the coordinate of the same name.

The scalar product projects out the component of $\mathbf{S}_i$ in the $\hat{\mathbf{z}}$ direction and we denote this by S_i^z.

$$H = -(h + qJ\sigma) \sum_i S_i^z \, .$$

$$Z_N = Z_1^N \, ,$$

where:

$$Z_1 = 2\pi \int_0^\pi \sin\theta \, d\theta e^{\beta(h+qJ\sigma)\cos\theta}$$

Make the change of variables:

$$\begin{aligned}
\cos\theta &= \mu, \quad \sin\theta \, d\theta = -d\mu, \\
Z_1 &= -2\pi \int_1^{-1} d\mu \, e^{\beta(h+q\sigma J)\mu} \\
&= 2\pi \int_{-1}^{1} d\mu \, e^{\beta(h+q\sigma J)\mu} \\
&= \frac{2\pi}{\beta(h+q\sigma J)} \left[e^{\beta(h+q\sigma J)} - e^{-\beta(h+q\sigma J)} \right] \\
&= \frac{4\pi}{\beta(h+q\sigma J)} \sinh[\beta(h+q\sigma J)] \, .
\end{aligned}$$

Mean value of spin:

$$\sigma = \frac{\partial \ln Z_1}{\partial(\beta h)} = \coth[\beta(h + q\sigma J)] - [\beta(h + q\sigma J)]^{-1}$$

For spontaneous magnetization, take $h = 0$, and consider

$$\sigma = \coth(a\sigma) - \frac{1}{a\sigma},$$

for $a = \beta q J$.

Plotting the two sides of the equation, we see that it has non-zero solutions when:

$$\frac{d}{d\sigma}\left[\coth(a\,\sigma) - \frac{1}{a\,\sigma}\right]_{\sigma=0} = \frac{a}{3} > 1\,.$$

Or when

$$T < T_c = qJ/3k\,.$$

10. The solution to this problem is essentially Section 7.8.

Q Solutions to Exercises H

1.

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)} \qquad (1); \qquad y' = \frac{y(x+y)}{(1+xy)} \qquad (2).$$

$$(x*, y*) = (0, 1) \qquad \text{Eqn}(1) \;\to 0 = \frac{0\times 1}{1} = 0 \qquad \text{eqn}(2) \to \qquad 1 = \frac{1(0+1)}{1+0} = 1$$

$$(x*, y*) = (0, 0) \qquad \text{Eqn}(1) \;\to 0 = \frac{0\times 1}{1} = 0 \qquad \text{eqn}(2) \to \qquad 0 = \frac{0\times 0}{1} = 0$$

$$x* = 1,\; 0 \le y* \le 1 \qquad \text{Eqn}(1) \qquad \to \qquad 1 = \frac{1(1+y)^2}{(1+y)^2} = 1 \qquad \forall y \qquad \text{eqn}(2) \Rightarrow y' = y.$$

$$x = 1 \leftrightarrow T = \infty;\; y = 0,\; B = \infty,\; y = 1,\; B = 0,$$

therefore ferromagnetic fixed point is $(0, 1)$.

Linearise (1) and (2) about $x' = x = 0,\; y' = y = 1$.

$$(1)\; x' = \left.\frac{x(1+1)^2}{(0+1)(1+0)}\right|_{y=1_{x\to 0}} = 4x - - - (a)$$

$$(2)\; y' = \left.\frac{y(0+y)}{1+0}\right|_{x=0_{y\to 1}} = y^2$$

Set

$$\begin{aligned} y' - 1 &= \delta y' \qquad \text{thus} \qquad y' = \delta y' + 1 \\ y - 1 &= \delta y \qquad \text{thus} \qquad y = \delta y + 1. \end{aligned}$$

and so

$$(2) \to \delta y' + 1 = (\delta y + 1)^2 = 2y\delta y + 1 + \delta(\delta y^2).$$

and

$$\delta y' = 2y\delta y/y = 1 = 2\delta y - - - (b).$$

From (a) and (b)

$$\begin{pmatrix} x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ \delta y \end{pmatrix} : \text{Eigenvalues } \lambda_1 = 4, \lambda_2 = 2.$$

From

$$\lambda_i = b^{yi}, \qquad b = 2; \quad \text{we have:} \quad y_1 = 2, \qquad y_2 = 1.$$

2.

$$\mathcal{Z} = \sum_{\{s\}} \Pi_{i=\ldots 2,4,6,\ldots} e^{\{KS_i(S_{i-1}+S_{i+1}) + \frac{\beta B}{2} S_{i-1} + \beta B S_i + \frac{\beta B}{2} S_{i+1} + 2C\}}.$$

Do the partial trace by summing over $S_2 = \pm 1, S_4 = \pm 1 \ldots$.

$$\begin{aligned} \mathcal{Z}' &= \sum_{\ldots S_1, S_3, S_5 \ldots} \Pi_{i=\ldots 2,4,6\ldots} \left[e^{\{K(S_{i-1}+S_{i+1}) + \beta B + \frac{\beta B}{2}(S_{i-1}+S_{i+1}) + 2C\}} \right. \\ &+ \left. e^{\{-K(S_{i-1}+S_{i+1}) - \beta B + \frac{\beta B}{2}(S_{i-1}+S_{i+1}) + 2C\}} \right] \end{aligned}$$

Re-label spins so that they are again numbered consecutively:

$$\begin{aligned} \mathcal{Z}' &= \sum_{\{S\}} \Pi_i \left[e^{\{(K+\frac{\beta B}{2})(S_i + S_{S+1}) + \beta B + 2C\}} \right. \\ &+ \left. e^{\{-(K-\frac{\beta B}{2})(S_i+S_{i+1}) - \beta B + 2C\}} \right] - - - (A). \end{aligned}$$

Require that new partition function should be same as old but with new coupling constants:

$$\mathcal{Z}' = \sum_{\{S\}} \Pi_i e^{\{K' S_i S_{i+1} + \beta B' S_i + C'\}} - - - (B).$$

Obviously, equations (A) and (B) are consistent if:

$$\begin{aligned} e^{\{K' S_i S_{i+1} + \beta B' S_i + C'\}} &= e^{\{(K+\frac{\beta B}{2})(S_i+S_{i+1}) + \beta B + 2C\}} \\ &+ e^{\{-(K-\frac{\beta B}{2})(S_i+S_{i+1}) - \beta B + 2C\}}. \end{aligned}$$

Equating exponents for $S_i, S_{i+1} = \pm 1$ yields 3 equations:

$$S_i, S_{i+1} = 1 : \qquad e^{K' + \beta B' + C'} = e^{2K + 2\beta B + 2C} + e^{-2K + 2C}.$$

$$S_i, S_{i+1} = -1 : \qquad e^{K' - \beta B' + C'} = e^{2K - 2\beta B + 2C} + e^{-2K + 2C}.$$

$$S_i = -S_{i+1} = \pm 1 : \qquad e^{-K' + C'} = e^{\beta B + 2C} + e^{-\beta B + 2C}.$$

These can be solved to yield:

$$e^{2\beta B'} = e^{2\beta B} \cosh(2K + \beta B) / \cosh(2K - \beta B).$$

$$e^{4K'} = \cosh(2K + \beta B) \cosh(2K - \beta B) / \cosh^2 \beta B.$$

$$e^{4C'} = e^{8C} \cosh(2K + \beta B) \cosh(2K - \beta B) \cosh^2 \beta B.$$

3. Given

$$K' = 2K^2 + L; \qquad L' = K^2,$$

at the fixed points we have

$$K^* = 2K^{*2} + L^*; \qquad L^* = K^{*2}. \tag{12}$$

It is easily verified that the fixed points are $(K^*, L^*) = (0,0), (\infty, \infty)$ and $(1/3, 1/9)$. The first two are the high-temperature and low-temperature points and are trivial. The non-trivial fixed point is $(K^*, L^*) = (1/3, 1/9)$ and we linearise about this. Set:

$$\begin{aligned} K' &= K^* + \delta K'; \qquad L' = L^* + \delta L' \\ K &= K^* + \delta K; \qquad L = L^* + \delta L. \end{aligned} \tag{13}$$

Eqns (12) become:

$$\begin{aligned} \delta K' &= 4K^*\delta K + \delta L + (2K^{*2} - K^* + L^*) \\ \delta L' &= 2K^*\delta K + (K^{*2} - L^{*2}). \end{aligned}$$

Hence:

$$\begin{pmatrix} \delta K' \\ \delta L' \end{pmatrix} = \begin{pmatrix} 4K^* & 1 \\ 2K^* & 0 \end{pmatrix} \begin{pmatrix} \delta K \\ \delta L \end{pmatrix} = \begin{pmatrix} 4/3 & 1 \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} \delta K \\ \delta L \end{pmatrix}.$$

$$\Rightarrow \text{eigenvalues } \lambda = \frac{1}{3}(2 \pm \sqrt{10})$$

therefore

$$\lambda_1 = 1.722; \ \lambda_2 = -0.390.$$

For the critical exponent ν we have

$\nu = 1/y$, where $\lambda_1 = b^{y_1}$, $b = \sqrt{2}$.

Hence

$$\nu = \frac{\ln b}{\ln \lambda_1} = \frac{\ln(\sqrt{2})}{\ln(1.722)} = 0.652.$$

4. Consider a lattice with probability p that any two sites are connected by a bond.

   Two or more sites connected = cluster, and the critical probability p_c is the probability of a cluster spanning the lattice.

   For $p < p_c$, probability of any two sites distance r apart being connected is $P(r)$ such that

   $$P(r) \to 0 \quad \text{as} \quad r \to \infty \quad \text{for} \quad p < p_c.$$

   Introduce correlation length by writing asymptotic form: $P(r) \sim l^{-r/\xi}$ as $r \to \infty$, where $\xi(p)$ = correlation length. Draw analogy between p in this model and temperature in Ising model. As $p \to p_c$, correlation length$\to \infty$, define critical exponent

   $$\xi \sim (p - p_c)^{\nu} \quad \text{as} \quad p \to p_c.$$

   Construct a new lattice by placing sites at centre of each unit cell of (original) p-lattice.

   Put bonds connecting any two sites not connected on p-lattice. New lattice is identical to old lattice but has bonds with probability

   $$q = 1 - p.$$

   Call the new lattice the q-lattice.

   If $p > p_c$, then at least one continuous path across the p-lattice but construction rules forbid this on the q-lattice.

   If $p > p_c$ then $q < q_i$ and conversely if $p < p_c$ then $q > q_c$. Therefore if the p-lattice is critical so also is the q-lattice. Therefore $q_c = 1 - p_c$ and as the lattices are identical, $q_c = p_c$. Hence $p_c = 1/2$.

   (a) Coarse-graining transformation. Remove alternate sites from the lattice and draw a new bond if there were at least two bonds connecting those particular sites on the old lattice.

(b) Re-scaling. Reduce all lengths by $b = \sqrt{2}$. New lattice is rotated $\pi/4$ to old one: new bonds lie along old diagonals.

Three configurations can contribute

- $p(a) = p^4$
- $p(b) = p^3(1-p)$
- $p(c) = p^2(1-p)^2$

and so

$$p' = p^4 + 4p^3(1-p) + 2p^2(1-p)^2 = 2p^2 - p^4.$$

At fixed point, $p' = p$,

RGE $\rightarrow p^4 - 2p^2 + p = 0$

or

$p(p-1)(p^2+p-1) = 0 \Rightarrow p^* = 0, 1\frac{-1\pm\sqrt{5}}{2}$.

We reject $\frac{-1\sqrt{5}}{2}$, as $p > 0$.

$p* = 0 \Rightarrow$ no bonds $p^* = 1 \Rightarrow$ all sites bonded.

Therefore trivial and analogous to high and low-temperature cases in thermal systems. Conclude critical fixed point is $p_c = p^* = (\sqrt{5}-1)/2 \simeq 0.62$.

At fixed point, RGE becomes

$$p^* = 2p^{*2} - p^{*4}.$$

Set $p' = p^* + \delta p'$, $p = p^* + \delta p$, then

$$\delta p' + p^* = 2(p^* = \delta p)^2 - (p^* + \delta p)^4 = 2p^{*2}(1 + \frac{\delta p}{p^*})^2 - p^{*4}(1 + \frac{\delta p}{p^*})^4$$

and to first order in δp

$$\delta p' = 4p^*(1 - p^{*2})\delta p \Rightarrow \delta p' = A_b(p^*)\delta p.$$

As this is a single-parameter space, there is only one eigenvalue, as A_b is a scalar,

$$\lambda_1 = 4p^*(1 - p^{*2})$$

and as $p^* = 0.62$, $\lambda_1 = 1.53$.

As $b = \sqrt{5}$, we have the critical index y_1 as

$$y_1 = \frac{\ln \lambda_1}{\ln b} = \frac{\ln 1.53}{\ln \sqrt{2}}$$

and recalling that $\nu = 1/y_1$,

$$\nu = \frac{\ln \sqrt{2}}{\ln 1.53} = 0.82.$$

R Solutions to Exercises I

1. Imposing the conditions for the fixed point on the given equations:

$$\begin{aligned} 0 &= 2\mu^* &+& \frac{\Lambda^2\lambda^*}{16\pi^2} + O(\epsilon^2) \\ 0 &= \epsilon\lambda^* &-& \frac{3\lambda^{*2}}{16\pi^2} \end{aligned}$$

Solving the second equation,

$$\lambda^* = 0 \text{ or } \lambda^* = \frac{16\pi^2}{3}\,\epsilon\,.$$

Substituting the second of these into the first equation

$$\mu^* = \frac{-\Lambda^2}{6}\,\epsilon + 0(\epsilon^2)\,,$$

Hence (μ^*, λ^*) is given by

$$\left(-\frac{\Lambda^2}{6}\,\epsilon, \quad \frac{16\pi^2}{3}\,\epsilon\right)\,,$$

as required.

To linearise, let

$$\begin{aligned}\lambda &= \lambda^* + \delta\lambda \\ \mu &= \mu^* + \delta\mu\end{aligned}$$

and with a few lines of routine algebra

$$\begin{aligned}\frac{d\,\delta\mu}{d\,\ln b} &= \left(2 - \frac{\epsilon}{3}\right)\delta\mu + \frac{\Lambda^2}{16\pi^2}\left(1 + \frac{\epsilon}{6}\right)\delta\lambda \\ \frac{d\,\delta\lambda}{d\,\ln b} &= -\epsilon\,\delta\,\lambda\,.\end{aligned}$$

Writing this in the form

$$\begin{pmatrix}\dfrac{d\delta\mu}{d\,\ln\delta} \\ \dfrac{d\delta\lambda}{d\,\ln b}\end{pmatrix} = \mathbf{A}_b\begin{pmatrix}\delta\mu \\ \delta\lambda\end{pmatrix}$$

$$\mathbf{A}_b = \begin{pmatrix} 2 - \epsilon/3 & \dfrac{\Lambda^2}{16\pi^2} \quad (1 + \frac{\epsilon}{6}) \\ 0 & -\epsilon \end{pmatrix}$$

From which we conclude:

$$y_1 = 2 - \frac{\epsilon}{3} > 0, \quad y_2 = -\epsilon < 0$$

Hence this is a **mixed** fixed point and determines critical behaviour for $\epsilon > 0$ and $d < 4$.

Critical exponent ν is obtained from

$$\nu = \frac{1}{y_1}$$

therefore

$$\begin{aligned}\nu = \frac{1}{2 - \epsilon/3} &= \frac{1}{2}\left[1 + \frac{\epsilon}{6} + O(\epsilon^2)\right] \\ &= \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2),\end{aligned}$$

as required.

2. Given:

$$
\begin{aligned}
p' &= b^2 p + c(b^2 - b^\epsilon)q + 0(pq)\,, \\
q' &= b^\epsilon q\,, \\
\epsilon &= 4 - d\,.
\end{aligned}
$$

Fixed point $(p^*, q^*) \equiv (0, 0)$.

Linearising about the fixed point, RGT transformation matrix is:

$$
\mathbf{A}_b = \begin{pmatrix} b^2 & c(b^2 - b^\epsilon) \\ 0 & b^2 \end{pmatrix}
$$

with eigenvalues and eigenvectors:

$$
\begin{aligned}
\lambda_1 &= b^2 & e^{(1)} &= \tbinom{1}{0} \\
\lambda_2 &= b^\epsilon & e^{(2)} &= \tbinom{-c}{0}\,.
\end{aligned}
$$

From $\lambda_i = b_i^y\,;\qquad y_1 = 2,\ y_2 = \epsilon\,.$

For a mixed (critical) fixed point in this case must have $y_2 = \epsilon < 0$: hence $d > 4$. Hence given fixed point is a critical point for lattice dimension greater than $d = 4$. Given information implies this model is in same universality class as the mean field theory of Ising. Hence indicates possible validity of the mean field theory above $d = 4$.