Solutions to Chapter 27 Exercises

in *Discrete Mathematics* by Norman L. Biggs;
2nd Edition 2002

27.1 The cycle index of a group of permutations

27.1.1

Use the method of trial and error to find all the distinguishable colourings of the corners of an equilateral triangle, assuming that three colours are available. (Regard the triangle as a flat piece of card, and assume that both sides are coloured.)

Solution There are just 10 distinguishable colourings:

$$\begin{array}{c}A\\A\quad A\end{array},\ \begin{array}{c}B\\B\quad B\end{array},\ \begin{array}{c}C\\C\quad C\end{array},\ \begin{array}{c}A\\B\quad B\end{array},\ \begin{array}{c}A\\C\quad C\end{array},$$

$$\begin{array}{c}B\\A\quad A\end{array},\ \begin{array}{c}B\\C\quad C\end{array},\ \begin{array}{c}C\\A\quad A\end{array},\ \begin{array}{c}C\\B\quad B\end{array},\ \begin{array}{c}A\\B\quad C\end{array}.$$

More systematically, there are 3 possibilities using only one colour, 6 using two colours, and 1 only using all three colours.

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27.1.3

Write down the cycle index of

(i) the group of symmetries of an equilateral triangle regarded as permutations of the corners,

(ii) the alternating group A_4,

(iii) the symmetric group S_5.

Solution

(i)
$$\frac{1}{6}\left(x_1^3 + 3x_1x_2 + 2x_3\right).$$

(ii)
$$\frac{1}{12}\left(x_1^4 + 8x_1x_3 + 3x_2^2\right).$$

(iii)
$$\frac{1}{120}\left(x_1^5 + 10x_1^3x_2 + 20x_1^2x_3 + 15x_1x_2^2 + 30x_1x_4 + 20x_2x_3 + 24x_5\right).$$

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27.2 Cyclic and dihedral symmetry

27.2.1

Write out in full the cycle indexes for C_{12}, D_{12}, and D_{14}.

Solution In this question it is assumed that the groups act on the corners of the relevant polygons. The answers are special cases of Theorems 27.2.1 and 27.2.2.

C_{12} : $\frac{1}{12}(x_1^{12} + x_2^6 + 2x_3^4 + 2x_4^3 + 2x_6^2 + 4x_{12})$;
D_{12} : $\frac{1}{12}(x_1^6 + x_2^3 + 2x_3^2 + 2x_6 + 3x_2^3 + 3x_1^2x_2^2)$;
D_{14} : $\frac{1}{14}(x_1^7 + 6x_7 + 7x_1x_2^6)$.

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27.2.2

Write down the cycle index for D_{2p} when p is an odd prime, and prove that

$$\zeta_{D_{2p}}(r, r, \ldots, r)$$

is an integer whenever r is a positive integer.

Solution According to Theorem 27.2.2,

$$\zeta_{D_{2p}} = \frac{1}{2p}(x_1^p + (p-1)x_p + px_1x_2^{p-1}).$$

Putting $x_1 = x_2 = x_p = r$, we get

$$\frac{1}{2p}(r^p + (p-1)r + pr^p) = \frac{1}{2p}\Big((p+1)r^p + (p-1)r\Big).$$

Fermat's theorem (p.149) tells us that $r^p - r$ is a multiple of p, say pA. So the expression can be written as

$$\frac{1}{2p}\Big((p+1)(r+pA) + (p-1)r\Big) = \frac{1}{2p}\Big(2pr + p(p+1)A\Big) = r + \frac{1}{2}(p+1)A,$$

which is an integer, since p is odd.

[See Theorem 27.4 for the significance of this result.]

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27.3 Symmetry in three dimensions

27.3.1

In Ex. 21.3.1 we showed that the rotational symmetry group of the regular tetrahedron is isomorphic to the alternating group A_4, that is, the group of even permutations of the corners. Show that every odd permutation of the corners corresponds either to a reflection of the tetrahedron, or to the composite of a reflection and a rotation.

Solution The odd permutations of the four corners are of two kinds, exemplified by: $(1\,2)$ and $(1\,2\,3\,4)$.

The first one corresponds to a reflection in a plane containing the edge 12 and passing through the centroid of the tetrahedron. For the second one, we have $(1\,2\,3\,4) = (1\,2)(2\,3\,4)$, the composite of the reflection $(1\,2)$ and the permutation $(2\,3\,4)$, which represents a rotation through $120°$ about the altitude passing though corner 1.

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27.3.2

Write down the cycle index of the rotational symmetry group of the tetrahedron, regarded as a group of permutations of the edges.

Solution If the vertices are $1, 2, 3, 4$ the six edges can be denoted by

$$a = 12,\ b = 13,\ c = 14,\ d = 23,\ e = 24,\ f = 34.$$

The permutations of the vertices correspond to the permutations of the edges as follows.

The identity corresponds to $(a)(b)(c)(d)(e)(f)$.
Eight rotations, such as (123) corresponding to $(adb)(cef)$.
Three rotations, such as (12)(34) corresponding to $(a)(be)(cd)(f)$.

Thus the cycle index is

$$\frac{1}{12}(x_1^6 + 8x_3^2 + 3x_1^2x_2^2).$$

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27.4 The number of inequivalent colourings

27.4.1

Use the table of cycle indexes given in Section 27.3 to write down formulae for the number of ways of colouring with r colours

(i) the corners of a tetrahedron;

(ii) the faces of a tetrahedron;

(iii) the faces of a cube.

Solution

(i) The group is simply the rotational symmetry group acting on the corners of the tetrahedron. So, by Theorem 27.4, the answer is

$$\frac{1}{12}\left(r^4 + 11r^2\right).$$

(ii) Here the group acts on the faces, but since the tetrahedron is self-dual the answer is the same.

(iii) Here the group acts on the faces of the cube, which (under duality, Fig. 27.9) correspond to the corners of the octahedron. Using the relevant cycle index from Table 27.3.2, the answer is

$$\frac{1}{24}\left(r^6 + 3r^4 + 12r^3 + 8r^2\right).$$

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27.4.3

In how many inequivalent ways can the faces of a regular dodecahedron be coloured red, white, or blue?

Solution Since the question refers to the *faces* of the dodecahedron, we need the cycle index of the group of the icosahedron, acting on its vertices. This is given in Table 27.3.2. Substituting $r = 3$ for all the variables gives

$$\frac{1}{60}\left(3^{12} + 15 \times 3^6 + 44 \times 3^4\right) = 9099\,.$$

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27.5 Sets of colourings and their generating functions

27.5.1

Write down the generating function for the set of colourings which assign the same colour to each member of a part of X, when $|X| = 10$ and

(i) there are three colours available, and X is partitioned into two equal subsets;

(ii) there are two colours available, and X is partitioned into subsets of size 4, 2, 2, 2.

Solution

(i) Here there are three colours a, b, c, and each part has size 5. According to Theorem 27.5 the generating function is

$$(a^5 + b^5 + c^5)(a^5 + b^5 + c^5) = (a^5 + b^5 + c^5)^2.$$

(ii) Here there are two colours a, b and the sizes of the parts are $4, 2, 2, 2$, so the generating function is

$$(a^4 + b^4)(a^2 + b^2)^3.$$

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27.6 Pólya's theorem

27.6.1

In how many ways can the faces of a cube be coloured so that there are two red faces, one white face, and three blue faces?

Solution As explained on p.396, the cycle index for the *faces* of the cube is the same as that for the *corners* of the octahedron. This is given in Table 27.3.2. Substituting $b^i + r^i + w^i$ for x_i, we require the coefficient of b^3r^2w in

$$\frac{1}{24}\Big[(b+r+w)^6 + 6(b+r+w)^2(b^4+r^4+w^4) + 3(b+r+w)^2(b^2+r^2+w^2)^2 + 6(b^2+r^2+w^2)^3 + 8(b^3+r^3+w^3)^2\Big].$$

Noting that only the first and third terms contain relevant powers, this coefficient is

$$\frac{1}{24}\left[\binom{6}{3,2,1} + 3\times 2\times 2\right] = \frac{1}{24}(60+12) = \frac{72}{24} = 3\,.$$

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27.6.2

Work out explicitly the generating function $U_D(b, g)$ for the number of ways of colouring the faces of a regular octahedron when the colours available are blue and green.

Solution Here it must be remembered that the *faces* of the octahedron are represented by the *corners* of a cube. Substituting $x_i = b^i + w^i$ in the cycle index given in Table 27.3.2, we obtain

$$\frac{1}{24}\Big[(b+g)^8 + 8(b+g)^2(b^3+g^3)^2 + 9(b^2+g^2)^4 + 6(b^4+g^4)^2\Big]$$

$$= \frac{1}{24}\Big[(b^8 + 8b^7g + 28b^6g^2 + 56b^5g^3 + 70b^4g^4 + 56b^3g^5 + 28b^2g^6 + 8bg^7 + g^8)$$
$$+ 8(b^2 + 2bg + g^2)(b^6 + 2b^3g^3 + g^6)$$
$$+ 9(b^8 + 4b^6g^2 + 6b^4g^4 + 4b^2g^6 + g^8)$$
$$+ 6(b^8 + 2b^4g^4 + g^8)\Big]$$

$$= (b^8 + g^8) + (b^7g + bg^7) + 3(b^6g^2 + b^2g^6) + 3(b^5g^3 + b^3g^5) + 7b^4g^4\,.$$

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27.6.4

When only two colours are involved it is often convenient to use the symbols 1 and x instead of b and w (or whatever names the actual colours may have). In this way we obtain a generating function of the form

$$f_0 + f_1 x + f_2 x^2 + \cdots + f_n x^n ,$$

where f_i is the number of configurations with i black objects and $n - i$ white ones. Work out this generating function explicitly for the necklace problem with 16 black or white beads, as discussed in the *Example*.

Solution The generating function is 1/32 times

$$(x+1)^{16} + 9(x^2+1)^8 + 2(x^4+1)^4 + 4(x^8+1)^2 + 8(x^{16}+1) + 8(x^2+1)^7(x+1)^2.$$

which is

$$1 + x + 8x^2 + 21x^3 + 72x^4 + 147x^5 + 280x^6 + 375x^7 + 440x^8 + \ldots .$$

(The coefficient of x^i in the remaining terms is the same as the given coefficient of x^{16-i}.)

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27.6.5

Chemists consider the kinds of molecule which can be formed from one C (carbon) atom linked to four radicals, each of which may be $HOCH_2$ (hydroxymethyl), C_2H_5 (ethyl), Cl (chlorine), or H (hydrogen). There are good reasons for using a picture of this situation in which the C atom is located at the centre of a regular tetrahedron, and the radicals occupy the corners.

(i) Show that there are 36 possible molecules.

(ii) Show that there are 15 molecules which contain no H radical.

(iii) Compute the generating function

$$H(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 ,$$

where h_i $(0 \leq i \leq 4)$ is the number of molecules which contain exactly i H radicals.

Solution The first two parts are simple applications of Theorem 27.4.

(i)

$$\frac{1}{12}(4^4 + 3 \times 4^2 + 8 \times 4^2) = \frac{1}{12}(256 + 48 + 128) = 36.$$

(ii)

$$\frac{1}{12}(3^4 + 3 \times 3^2 + 8 \times 3^2) = \frac{1}{12}(81 + 27 + 72) = 15.$$

(iii) We have the results $h_0 = 15$ and $h_0 + h_1 + h_2 + h_3 + h_4 = 36$. Simple arguments show that $h_4 = 1$, $h_3 = 4$. So it remains only to compute one more coefficient, say h_2. If exactly two radicals are H, the other two are either the same (3 possibilities) or different (also 3 possibilities). Thus we require 3 times the sum of the coefficients of a^2b^2 and a^2bc in

$$\frac{1}{12}\big((a+b+c)^4 + 8(a+b+c)(a^3+b^3+c^3) + 3(a^2+b^2+c^2)^2\big).$$

By inspection, both coefficients are 1, hence so $h_2 = 6, h_1 = 10$ and

$$H(x) = 15 + 10x + 6x^2 + 4x^3 + x^4.$$