Introduction

A modern approach to the existence and regularity theory of partial differential equations relies on obtaining suitable a priori estimates in terms of the information available on the data, typically in the form of norms in appropriate functional spaces. Following such ideas, this book is devoted to obtaining basic estimates for some particular nonlinear parabolic equations and to derive consequences about qualitative and quantitative aspects of the theory. In pursuing this aim, a major role is given to the scaling properties and the existence of suitable self-similar solutions; results of symmetrization and mass concentration comparison also play a prominent role; finally, a strategy of looking for the worst case completes the picture.

Using this machinery, we can derive a complete set of a priori estimates in Lebesgue and Marcinkiewicz spaces for the two model equations of nonlinear diffusion theory:

$$u_t = \Delta u^m, \quad u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \tag{1}$$

which we study for the different values of the exponents m and p and space dimension $n \geq 1$. Both equations are popular models in that area, with a number of applications in physics and other sciences, and with a very rich mathematical theory. We will give preference in the presentation to the first equation, which is known in the literature as the porous medium equation (shortly, PME) when $m > 1$, and the fast diffusion equation (FDE) when $m < 1$. The classical heat equation (HE) is included as the case $m = 1$, but contrary to the standard approach for the latter equation, our approach here is heavily nonlinear in methods and results. The bulk of the book is devoted to the analysis of smoothing estimates and the related topics that arise for the porous medium equation and its relative, the fast diffusion equation.

The second equation mentioned above is usually called the p-Laplacian evolution equation, and is the best known of a series of models of nonlinear parabolic equations called "gradient-dependent diffusion equations". Our interest in including it here, even if with a lower level of attention, serves the purpose of showing that the systematic study of the PME / FDE is applicable to the p-Laplacian equation, and also to a number of usual models that appear all the time in the theory and applications of nonlinear diffusion, and more generally, in reaction diffusion.

The tools

Let us review the technical tools: the topics of symmetrization and mass concentration comparison relevant for our study have been explained in detail in the paper [Va04b], which was focused on a more general model, the so-called filtration equation,

$$u_t = \Delta\varphi(u) + f, \tag{2}$$

where usually φ is a C^1 real function with positive derivative; in degenerate cases φ' is only non-negative; the equation may also include singularities, i.e., values where φ is not C^1 and $\varphi'(u) = \infty$. Equation (2) includes as a particular case our main model, $u_t = \Delta u^m$, as well as other popular nonlinear diffusion models, like the Stefan equation, where $\varphi(u) = (u - 1)_+$. Our results in this paper have corollaries for such equations, though we will not develop them. Many nonlinear diffusion variants can also be treated by similar methods, like the already mentioned p-Laplacian equation, and we will give details of this extension.

On the other hand, the model equations we consider have the extra property of being invariant under a large group of scaling transformations, with at least two free parameters (see formula (1.22) below). Indeed, the property of scale invariance is the connecting link between the two otherwise quite different parabolic equations (1). Moreover, both equations can be combined into a more complicated model, so-called doubly nonlinear equation, to which the methods apply. A striking consequence of scale invariance is the existence of self-similar solutions whose behaviour and properties can be described in great detail. Special cases of such solutions will serve as model examples in our worst-case strategy to obtain estimates. They are the main stars of our show.

Let us point out that obtaining particular solutions of PDEs does not seem a fundamental problem in the most theoretical approach to the subject, but reality is different under such a deceptive cover: self-similarity, separation of variables, the Bäcklund transformation, the method of characteristics, Green's function, integral transforms, and other methods allow the applied mathematician to gain the insight that serves as a corner-stone for the general treatment. This approach will lead our steps in what follows.

The goal. Smoothing

The present work uses the approach we have outlined to derive a complete set of a priori estimates that should play an important role in the qualitative theory of the equations. Let us present the motivation for this application, starting from the classical heat equation, $u_t = \Delta u$. This equation has a remarkable property, called *smoothing*, whereby solutions with (initial and boundary) data in suitable functional spaces are actually C^∞ smooth functions in the interior of the domain of definition. This is a result easily obtained in the case of the Cauchy problem posed in the whole space by using the standard representation with the Gaussian kernel. It was soon observed that the property is shared by a whole class of so-called linear parabolic equations with variable, even non-smooth coefficients, being remarkable in this context the work of J. Nash [Na58, KN02], while the elliptic counterparts bear the names of E. De Giorgi [DG57] and J. Moser [Mo61]. The results were later extended to wide classes of nonlinear equations under suitable structural assumptions, the main ones being uniform parabolicity (or ellipticity) and smoothness of the coefficients. Uniform parabolicity will be missing in this text.

When performing the regularity proofs in more general contexts, one is led to proceed by steps: first, data in general spaces, like $u_0 \in L^p$ spaces, produce solutions which are bounded, $u \in L^\infty$, and this is the starting point of the 'improvement process'; in a second step, bounded solutions are shown to be continuous, usually Hölder continuous, by means of a priori interior bounds in terms of the proven bounds for the solution; third, an iterative argument allows us to obtain estimates for first derivatives and then derivatives of all orders. When dealing with increasingly wider classes of equations, mainly nonlinear equations or equations with bad coefficients, the latter steps may or may not be true (a phenomenon called partial regularity), and the first step receives special attention, being the most general. This is why we will accept some usual terminology in nonlinear PDE analysis and give the (rather incorrect) name of *smoothing effect* to the property which could be better termed 'function space improvement' and says: data in a space like $L^p(\mathbf{R}^n)$ produce solutions that live for $t > 0$ in a space $L^q(\mathbf{R}^n)$ with $q > p$, hopefully $L^\infty(\mathbf{R}^n)$. Describing when the aforementioned equations do or do not possess such a smoothing effect, what are the quantitative estimates behind that property, or what happens otherwise is the main purpose of this text.

Similar estimates are well known in the theory of linear equations and semigroups. Semigroups that send initial data in $L^1(\Omega, dx)$ into orbits in $L^\infty(\Omega, dx)$ are called ultracontractive, see [Dv89, Chapter 2], and this is the type of property of concern for us. But the reader should notice that while for linear equations any boundedness estimate is equivalent to a stability result (i.e., control of differences of solutions in terms of differences of data), this is not at all the case for nonlinear semigroups. Since the equation $u_t = \Delta u^m$ is nonlinear for $m \neq 1$, the estimates that give boundedness of the PME/FDE flow do not necessarily imply any kind of contractivity or stability, and this is a main source of divergence in the corresponding theory. We will return to this issue in Section 4.2.

Semigroups, bounds, asymptotic rates, and patterns

There are many sides in this task under the flag of nonlinear evolution. Thus, from the point of view of functional analysis, the whole question of smoothing effects can be seen as a rather basic part of the study of the evolution semigroups generated by the equations, as was pointed out in the pioneering works of Bénilan [Be72, Be76] and Véron [Ve79]. As we will see, it gives in some cases the information needed to properly define the semigroup in a suitable domain; in any case, it shows how the semigroup behaves in time.

The text also contributes to the topic of asymptotic behaviour. Given an equation E, the idea is to associate to every data, in our case initial data u_0, a set of so-called 'asymptotic data' that allow us to reconstruct the long-time behaviour of the solution generated by E from u_0. In the standard application the solution of a nonlinear diffusion process exists globally in time and goes to zero in a more or less uniform way. The asymptotic data must be simple to calculate and must allow us to reconstruct the approximate behaviour of $u(x, t)$ for large times (so-called *intermediate asymptotics*); they take the form of *asymptotic rates* (which are usually powers of time), as well as the spatial pattern, usually called the *asymptotic profile*. We have contributed other studies on the asymptotic behaviour of the PME [Va03, Va04]. Some of the main new contributions of our text lie in the realm of fast diffusion and are related to extinction phenomena. By this we mean that for some evolution equations and data the solution disappears completely at a certain time $T > 0$ in the sense that $u(x, t) \to 0$ at all points $x \in \mathbf{R}^n$ as $t \nearrow T$.

This work aims at giving the reader a sound knowledge of the behaviour of the solutions of the semigroups generated by the two families of equations (1), specially the first one. The idea of comparison with the well-known heat equation and its Gaussian kernel is always present. The similarities and striking differences will be outlined. Our main characters will be the self-similar functions that play the role of the Gaussian kernel in the different contexts, and the plot will almost always revolve around them.

Contents and distribution

The analysis of the results based on comparison with source-type solutions occupies Part I and concerns the PME, the HE and the supercritical range of the FDE, $m > m_c$ with $m_c = (n - 2)/n$. A complete description of the L^p–L^q effects is produced, and some natural spaces are introduced: the space $\mathcal{M}$ of finite measures and the Marcinkiewicz M^p or weak L^p spaces. A number of topics of general interest is discussed, as indicated in the introduction to that part.

Part II deals with the critical and subcritical range of the FDE, $m \le m_c$. This study offers a large number of novelties, like extinction, backward effects, and delayed boundedness. These novel situations show the sharp contrast between linear and nonlinear equations that occurs in different instances and affects all the basic questions of the theory.

A complete summary of the results obtained so far is contained in Chapter 10. This chapter also summarizes the evolution of Dirac masses in the different exponent ranges, which very much reflects the variation with m of the diffusive power of the equation.

Once the analysis of the PME/FDE has been completed, we devote Part III to extensions and appendices.

As for the first, Chapter 11 contains the application of the same strategies to obtain estimates for the other equations, the p-Laplacian equation and the doubly nonlinear equation. The aim is illustrative of the scope of the method, hence the study of the extension is shorter and less complete. The treatment is necessarily sketchy.

A series of three appendices contains important technical material that is needed but is not in the main line of the book. We hope they will be useful for the reader. We add a final section containing comments and bibliographical notes and end with a brief comment on extensions.

Preference is given to the Cauchy problem, for a question of definiteness, simplicity, and space. This restriction allows us to build a rather complete theory. However, some hints about Dirichlet, Neumann, or local solutions are reflected here and there.

We refer to the monograph [V06] for further details on these problems and also on the peculiarities of solutions with changing sign.

We include at the end of each chapter some additional information on the contents of the different sections, main references, and possible developments, as an orientation for the reader. In doing that we have tried to be concise and fair, and not stray far away from the main subject, but we allow ourselves diversions that might be appealing for the expert or the curious reader, and, of course, we follow our own taste in the matter. We ask for apologies if unintended omissions of important topics arise. Most chapters contain a list of exercises.

Note In obtaining our estimates we will restrict our attention to non-negative solutions and data. Signed solutions can be considered but then the equation must be written as $u_t = \Delta(|u|^{m-1}u)$, or even better $u_t = \nabla \cdot (|u|^{m-1}\nabla u)$ after scaling out the constant m. Restriction to non-negative data is done mostly for convenience, since on the one hand many results will be directly applicable to signed solutions once the non-negative case is settled by using the maximum principle; on the other hand, the physical applications deal in general with the situation $u \geq 0$.

Reading and using the book

The book is aimed at providing an introduction to the subject of smoothing estimates for nonlinear diffusion equations, centred in the study of some model equations and on the Cauchy problem, which is mathematically the most natural in that respect. At the same time, it gives a rather comprehensive account of the state of the art in the case of the PME/FDE, specially the latter one whose theory is still being developed. Consequently, it should be useful as an advanced graduate textbook and also as a source of information for graduate students and researchers. In any case, we have to point out that nonlinear diffusion is a vast field, so that having a text of a manageable size implies a strict selection of material. Many interesting topics and connections have been necessarily omitted for that reason if they played no essential role in the picture we wanted to develop. Some of them are briefly touched on in the sections entitled Comments and historical notes.

As a subject for a PhD course, there are a number of selections depending on purpose and taste. One such selection centres around the supercritical areas contained in Chapters 2 and 3, and some subcritical topics of Chapters 5 *et seq*, plus the inclusion of the p-Laplacian extensions. This option can also be complemented with estimates obtained by an energy approach or other methods as often done in [V06].

Another option is to go as quickly as possible to Chapter 5 and then concentrate on the mysteries of subcritical fast diffusion. There, extensions for further study come from the cited literature.

A basic selection of the text can be combined with advanced topics in related areas. An example is the basic PME theory developed in [V06]. It can be complemented with cases of the dynamical systems approach of [GV03]. Free-boundary problems, kinetic equations and geometrical problems are other options.

As prerequisites, the author has in mind PhD students having followed courses in classical analysis, functional analysis, ODEs and PDEs. Knowing some physics of continuous media or studying the subject in parallel is useful and recommended, but not required. Though the basic part of the text is self-contained, material from other sources is used in the advanced sections that are meant to introduce current problems; in those cases, the pertinent references are carefully indicated and will hopefully serve to introduce the reader to the specialized literature.

Part I
Estimates for the PME/FDE

The first chapter provides necessary preliminaries on functional analysis, comparison results, and the fundamentals of the PME. Additional information is supplied in the appendices at the end.

Chapter 2 discusses the smoothing and decay effects for the porous medium equation, using as a model case the famous Barenblatt solutions that have explicit formulas. This material has been the foundation for all later developments. Important issues are introduced and discussed at length, like scaling techniques, optimal decay, best constants, and the distinction between strong and weak smoothing effects.

Chapter 3 covers the smoothing effects that arise from comparison in Marcinkiewicz spaces, our second major topic. Weak L^p–L^q effects are discussed in Section 3.6.

The short Chapter 4 contains lower estimates (positivity estimates) and Harnack inequalities. It also deals with contractivity, error estimates, and continuous dependence.