1

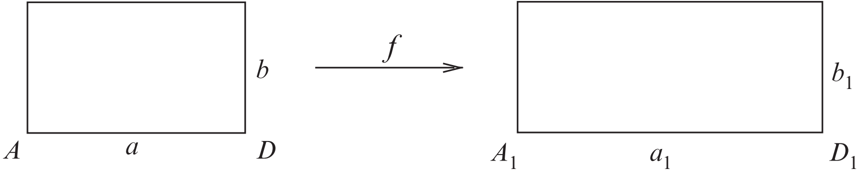
The Grötzsch Argument

In this chapter, we use Grötzsch's problem of finding a homeomorphism between two rectangles which is closest to a conformal map to motivate the idea of quasiconformality for C^1-diffeomorphisms. We will see that the affine stretching is our first example of a quasiconformal map which is not conformal. We also include a short summary of the basic theory of Riemann surfaces.

1.1 Maps on rectangles

A mapping f on a connected plane domain Ω is conformal, if f is a homeomorphism of Ω and f is holomorphic in Ω. If Ω has a smooth boundary away from finitely many corners, and is simply connected, fix three points A, B, and C on $\partial\Omega$, and fix three points A_1, B_1, and C_1 on $\partial\mathbb{D}$, then there exists a unique $f : \Omega \to \mathbb{D}$ and f extends continuously to $\partial\Omega$ with $f(A) = A_1$, $f(B) = B_1$ and $f(C) = C_1$. This is one formulation of the Riemann mapping theorem.

Let R be a rectangle with vertices prescribed in clockwise order A, B, C, D and side lengths a and b, and also let R_1 be a rectangle with corresponding vertices A_1, B_1, C_1, D_1 and side lengths a_1 and b_1, and consider $f \in C^1$, where C^1 is the space of continuously differentiable functions, such that $f : R \to R_1$ and $f(A) = A_1$, $f(B) = B_1$ and $f(C) = C_1$.



A natural question to ask is whether f can be conformal. One can try the affine map

$$A_f : (x, y) \mapsto \left(\frac{a_1 x}{a}, \frac{b_1 y}{b}\right),$$

which maps R onto R_1, but this is not conformal unless $a/b = a_1/b_1$. In fact, there is only a conformal map from R to R_1 if $a/b = a_1/b_1$.

To see this, assume $f : R \to R_1$ is conformal with the required properties. Then, by the reflection principle, f can be extended to a conformal map from $\mathbb{C}$ to $\mathbb{C}$.

Exercise 1.1.1. Show that this means that f must be of the form $f(z) = \mu z + \nu$ for some $\mu, \nu \in \mathbb{C}$ and hence that $a/b = a_1/b_1$.

Grötzsch [25] asked, what is the map from R to R_1 closest to a conformal map which satisfies the above properties? Before answering this question, we need to make precise what is meant by being closest to a conformal map.

1.2 Some definitions

Let $f : \Omega \to f(\Omega)$ be a C^1-diffeomorphism, and $z = x + iy$ is a coordinate on Ω at z_0 and $w = u + iv$ is a coordinate on $f(\Omega)$ at $f(z_0)$. Then f induces a linear mapping between the corresponding tangent spaces, and the differential defines the linear map given by

$$\begin{aligned} \mathrm{d}u &= u_x \mathrm{d}x + u_y \mathrm{d}y, \\ \mathrm{d}v &= v_x \mathrm{d}x + v_y \mathrm{d}y. \end{aligned}$$

This can be summarized in complex form

$$\mathrm{d}w = \mathrm{d}f = f_z \mathrm{d}z + f_{\overline{z}} \mathrm{d}\overline{z},$$

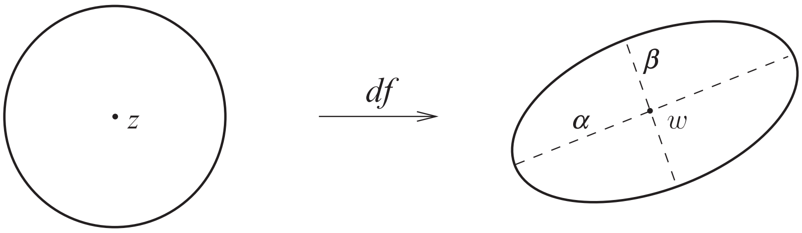
where the complex derivatives are given by

$$\begin{aligned} f_z &= \frac{1}{2}(f_x - i f_y) \\ f_{\overline{z}} &= \frac{1}{2}(f_x + i f_y) \end{aligned}$$

Since f is a diffeomorphism, it is well approximated by a linear map (in this case at z_0). The linear map df maps the unit circle in the z-plane onto an ellipse with major axis of length α and minor axis of length β in the w-plane.

The area of $\mathbb{D}$ is π, and the area of $df(\mathbb{D})$ is $\pi\alpha\beta$, so the Jacobian $J_f(z_0)$ is $\alpha\beta$.

Exercise 1.2.1. Show that $J_f = |f_z|^2 - |f_{\overline{z}}|^2$.



Since sense-preserving maps are being considered here, $|f_z| > |f_{\overline{z}}|$, and so $J_f > 0$. The dilatation (or distortion) at z_0 is defined to be

$$D_f := \frac{\alpha}{\beta} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1.$$

The complex dilatation at z_0 is

$$\mu_f = \frac{f_{\overline{z}}}{f_z}.$$

The dilatation and distortion are related by

$$D_f = \frac{1 + |\mu_f|}{1 - |\mu_f|}.$$

By the Cauchy–Riemann equations, if $f \in C^1$, then f is conformal implies that $f_{\overline{z}} \equiv 0$. If f is conformal, $D_f = 1$ and $\alpha = \beta$ above, so df maps circles to circles.

Definition 1.2.1. *Let $f : \Omega \to \mathbb{C}$ be a diffeomorphism. We say that f is a quasiconformal map if $D_f(z)$ is bounded in Ω. We say f is a K-quasiconformal map if $D_f(z) \leq K$ for all $z \in \Omega$. K_f is defined to be the infimum of $K \geq 1$ for which f is K-quasiconformal.*

Every C^1-diffeomorphism is K-quasiconformal on every compactly contained open set in Ω for some K.

Exercise 1.2.2. Show that $K_f = \sup_{z \in \Omega} D_f(z)$. Show that if g and h are conformal mappings, and $\widetilde{f} = h \circ f \circ g$, then $D_f(z) = D_{\widetilde{f}}(g^{-1}(z))$.

A C^1-diffeomorphism is conformal if and only if it is 1-quasiconformal, and so, in some sense, the larger K_f is, the further f is from a conformal map.

1.3 Solving the Grötzsch problem

Now we are in a position to say what is meant by being closest to conformal. Recalling the notation of Section 1.1, we want to know if there is a map f which maps R into R_1, preserving vertices, such that $K_f \leq K_g$ for any other homeomorphism $g : R \to R_1$ with the same properties, and if so, what is K_f?

Exercise 1.3.1. Show that the affine map A_f considered earlier has the following explicit form:

$$A_f(z) = \frac{1}{2}\left(\frac{a_1}{a} + \frac{b_1}{b}\right) z + \frac{1}{2}\left(\frac{a_1}{a} - \frac{b_1}{b}\right)\overline{z}. \tag{1.1}$$

Thus, we get

$$(A_f)_z = \frac{1}{2}\left(\frac{a_1}{a} + \frac{b_1}{b}\right),$$
$$(A_f)_{\overline{z}} = \frac{1}{2}\left(\frac{a_1}{a} - \frac{b_1}{b}\right),$$

and the dilatation of A_f is constant,

$$D_{A_f}(z) = \frac{a_1 b}{b_1 a}.$$

Now consider an arbitrary f with the required properties, and a horizontal path $\gamma \in R$. Then

$$\begin{aligned} a_1 \leq \text{length of } f(\gamma) &= \int_0^a |f_z \mathrm{d}z + f_{\overline{z}}\mathrm{d}\overline{z}| \\ &= \int_0^a |f_z| \left| 1 + \frac{f_{\overline{z}}}{f_z} \right| \mathrm{d}x \end{aligned}$$

because γ is a straight line.

$$\Rightarrow a_1 \leq \int_0^a |f_z||1+\mu_f|\mathrm{d}x.$$

Now integrate over R.

$$\begin{aligned} a_1 b &\leq \int\int_R |f_z||1+\mu_f|\mathrm{d}x\,\mathrm{d}y \\ &= \int\int_R \frac{|f_z||1+\mu_f|\sqrt{J_f}}{\sqrt{|f_z|^2 - |f_{\overline{z}}|^2}}\mathrm{d}x\,\mathrm{d}y \\ &= \int\int_R \frac{|1+\mu_f|}{\sqrt{1-|\mu_f|^2}}\sqrt{J_f}\mathrm{d}x\,\mathrm{d}y \\ &\leq \left(\int\int_R \frac{|1+\mu_f|^2}{1-|\mu_f|^2}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \left(\int\int_R J_f\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \end{aligned}$$

using the L^2 version of the Cauchy–Schwarz inequality. Now, since J_f is the area of $f(R)$,

$$\frac{a_1 b^2}{b_1} \leq \int\int_R \frac{|1+\mu_f|^2}{1-|\mu_f|^2}\mathrm{d}x\,\mathrm{d}y \tag{1.2}$$

From (1.2),

$$\begin{aligned}\frac{a_1 b^2}{b_1} &\leq \int\int_R D_f \,\mathrm{d}x\,\mathrm{d}y\\ \Rightarrow \frac{a_1 b}{b_1 a} &\leq K_f.\end{aligned}$$

So the affine map A_f gives the smallest K_f, i.e., if $f : R \to R_1$, then $K_f \geq K_{A_f}$.

Exercise 1.3.2. Show that the above affine map given by (1.1) is the only such map from R to R_1 which has dilatation $(a_1 b)/(b_1 a)$. (Hint: all the inequalities above must be equalities.)

1.4 Composed mappings

Let $f : \Omega \to f(\Omega)$ and $g : f(\Omega) \to \mathbb{C}$, with z a coordinate on Ω and ζ a coordinate on $f(\Omega)$.

Exercise 1.4.1. Show that

$$\overline{(f_z)} = (\overline{f})_{\overline{z}}$$

and

$$\overline{(f_{\overline{z}})} = (\overline{f})_z.$$

Now show that the complex derivatives of a composed mapping $g \circ f$ are

$$\begin{aligned}(g \circ f)_z &= (g_\zeta \circ f) f_z + (g_{\overline{\zeta}} \circ f)\overline{(f_{\overline{z}})},\\ (g \circ f)_{\overline{z}} &= (g_\zeta \circ f) f_{\overline{z}} + (g_{\overline{\zeta}} \circ f)\overline{(f_z)}.\end{aligned} \tag{1.3}$$

Solving these equations for $g_\zeta \circ f$ and $g_{\overline{\zeta}} \circ f$ gives

$$\begin{aligned}g_\zeta \circ f &= \frac{1}{J_f}\Big((g \circ f)_z \overline{(f_{\overline{z}})} - (g \circ f)_{\overline{z}}\overline{(f_z)}\Big)\\ g_{\overline{\zeta}} \circ f &= \frac{1}{J_f}\left((g \circ f)_{\overline{z}} f_z - (g \circ f)_z f_{\overline{z}}\right),\end{aligned} \tag{1.4}$$

where J_f is the Jacobian of f (recall Exercise 1.2.1). Now let $g = f^{-1}$, then from (1.4) we get

$$\begin{aligned}(f^{-1})_\zeta \circ f &= \frac{\overline{(f_z)}}{J_f},\\ (f^{-1})_{\overline{\zeta}} \circ f &= -\frac{f_{\overline{z}}}{J_f},\end{aligned}$$

and so

$$\mu_{f^{-1}} \circ f = \left(-\left(\frac{f_z}{|f_z|} \right)^2 \mu_f \right).$$

From (1.4), we get

$$\mu_{g\circ f} = \frac{\mu_f + r_f(\mu_g \circ f)}{1 + r_f \overline{\mu_f}(\mu_g \circ f)}, \tag{1.5}$$

where $r_f = \overline{f_z}/f_z$. If g is conformal,

$$\mu_{g\circ f} = \mu_f, \quad \text{and} \quad D_g \circ f = D_f.$$

If f is conformal,

$$\mu_{g\circ f} = (\mu_{g\circ f}) \frac{\overline{(f_z)}}{f_z}, \quad \text{and} \quad D_{g\circ f} = D_g.$$

Finally, if $h = g \circ f$,

$$\mu_{h\circ f^{-1}} \circ f = \left(\frac{f_z}{\overline{(f_z)}} \right) \frac{\mu_h - \mu_f}{1 - (\overline{\mu_f})\mu_h}. \tag{1.6}$$

Theorem 1.4.1. *If f is K-quasiconformal, and g is K_1-quasiconformal, then $f \circ g$ and $g \circ f$ are both KK_1-quasiconformal maps.*

Proof. As a shorthand, g_ζ will here be used to denote $g_\zeta \circ f$. Now the dilatation of $g \circ f$ is given by

$$\begin{aligned} D_{g\circ f} &= \frac{|(g\circ f)_z| + |(g\circ f)_{\bar z}|}{|(g\circ f)_z| - |(g\circ f)_{\bar z}|} \\ &= \frac{|g_\zeta f_z + g_{\bar\zeta}\overline{f_{\bar z}}| + |g_\zeta f_{\bar z} + g_{\bar\zeta}\overline{f_z}|}{|g_\zeta f_z + g_{\bar\zeta}\overline{f_{\bar z}}| - |g_\zeta f_{\bar z} + g_{\bar\zeta}\overline{f_z}|} \\ &\le \frac{|f_z||g_\zeta| + |f_z||g_{\bar\zeta}| + |f_{\bar z}||g_\zeta| + |f_{\bar z}||g_{\bar\zeta}|}{|f_z||g_\zeta| + |f_z||g_{\bar\zeta}| - |f_{\bar z}||g_\zeta| - |f_{\bar z}||g_{\bar\zeta}|} \\ &= D_f D_g. \end{aligned}$$

The calculation runs analogously for $f \circ g$. ☐

Example 1.4.1. Let z be given by $re^{i\theta}$ in polar coordinates, and consider the mapping $w = f(z) = r^t \mathrm{e}^{i\theta}$, where $t > 0$ is a fixed parameter. Despite the fact that f is not bi-Lipschitz near 0, f is actually a quasiconformal mapping of $\mathbb{C}$. We see that

$$\log w = t \log r + i\theta$$

and so, locally, f is of the form conformal$\circ$affine$\circ$conformal, where the affine mapping is a stretching by a factor t. Therefore f is a K-quasiconformal mapping of $\{0 < |z| < \infty\}$ onto $\{0 < |w| < \infty\}$, where $K = \max\{t, 1/t\}$. Since $z = 0$ is an isolated singularity, we will see later that it is removable, and so f is a K-quasiconformal mapping of $\mathbb{C}$ onto itself. The complex dilatation is, for $z \neq 0$,

$$\mu_f(z) = \frac{t-1}{t+1}\frac{z}{\overline{z}}$$

and so

$$\sup_{z\in\mathbb{C}} |\mu_f(z)| = \left|\frac{t-1}{t+1}\right| .$$

1.5 Riemann surfaces

In this section, we present some of the basic facts about Riemann surfaces. A real n-dimensional manifold M is a Hausdorff space with a countable base for topology which is locally homeomorphic to the standard euclidean space $\mathbb{R}^n$. In other words, for all $p \in M$, there exists an open neighbourhood $U_p \subset M$ of p and a homeomorphism π_p from U_p onto an open set in $\mathbb{R}^n$. The pair (U_p, π_p) is called a coordinate chart at p. If two coordinate charts (U_p, π_p) and (U_q, π_q) overlap, so that $U_p \cap U_q$ is non-empty, then the map $\pi_q \circ \pi_p^{-1}$ defined on $\pi_p(U_p \cap U_q)$ from an open set of $\mathbb{R}^n$ into $\mathbb{R}^n$ must be a homeomorphism. If there is a collection of charts $\{(U_{p_i}, \pi_{p_i})\}$ for which the union of their domains is the whole of M, then the collection is called an atlas on M.

The complex space $\mathbb{C}^n$ of dimension n can be viewed as the real space $\mathbb{R}^{2n}$ of dimension $2n$ via the identification

$$(x_1 + iy_1, \ldots, x_n + iy_n) \longleftrightarrow (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

and so a complex manifold of dimension n can be viewed as a real manifold of dimension $2n$. A complex manifold is called analytic if all the transition maps $\pi q \circ \pi_p^{-1}$ from $\pi_p(U_p \cap U_q)$ into $\mathbb{C}^n$ are biholomorphic.

Definition 1.5.1. *A Riemann surface is a one-dimensional complex analytic manifold.*

The analytic structure of a Riemann surface is also called a conformal structure, since the transition maps between charts are, in this case, always conformal. Analytic functions can be defined between Riemann surfaces in the following way. A function $f : M \to N$ between Riemann surfaces is analytic at $p \in M$ if there exists coordinate charts π_p on M and $\pi_{f(p)}$ on N such that $\pi_{f(p)} \circ f \circ \pi_p^{-1}$ is holomorphic at $\pi_p(p)$. This is invariant under transition maps between coordinate charts. We then say that $f : M \to N$ is analytic if it is analytic at each $p \in M$.

A collection φ of complex functions φ_i defined on open sets U_i is called an (m, n)-differential on M if

$$\varphi_i \left(\frac{\mathrm{d}\pi_i}{\mathrm{d}\pi_j}\right)^m \left(\overline{\frac{\mathrm{d}\pi_i}{\mathrm{d}\pi_j}}\right)^n = \varphi_j$$

on $U_i \cap U_j$. The differential φ is called holomorphic if each φ_i is holomorphic. Important cases are quadratic differentials, for which $m = 2$ and $n = 0$, and Beltrami differentials, for which $m = -1$ and $n = 1$. We now consider some properties of general surfaces.

Definition 1.5.2. *A smooth covering surface of a surface M, is a pair (S, f) where S is a surface and $f : S \to M$ is a local homeomorphism.*

The map f is a projection. A smooth covering surface (S, f) of M is called regular if every path in M has a lift to S from each point lying over the initial point. In this case, f is surjective.

Theorem 1.5.1 (Monodromy theorem). *Let (S, f) be a regular covering surface of M and γ_0, γ_1 be homotopic paths in M. Then the lifts of γ_0, γ_1 on S from the same initial point have the same end point and are homotopic.*

If a covering surface S is simply connected, then S is called the universal covering surface. The universal covering surface is the maximal covering surface, in the sense that is S' is any other covering surface of M, then S covers S'. Every surface has a universal covering surface.

If (S, f) is a smooth covering surface of a surface M, then a cover transformation g of S over M is a homeomorphism $g : S \to S$ such that $f \circ g = f$. The collection of all such mappings g form a group G under composition, which is called the covering group of S over M. If the projection $f : S \to M$ is surjective and the covering group G of S over M is transitive, then the quotient surface S/G is homeomorphic to M.

Every point of a covering surface S has a neighbourhood in which no two points are equivalent under the action of the covering group, and except for the identity, no covering transformation has any fixed points. The covering group of a universal covering surface S over M is transitive, and so for a universal covering surface, S/G is always homeomorphic to M.

Now let M be a Riemann surface and (S, f) be a smooth covering surface of M. Then S carries a unique conformal structure which makes f analytic. This implies that the covering transformations of S over M are conformal. The following remarkable theorem classifies Riemann surfaces.

Theorem 1.5.2 (Riemann mapping theorem). *Every simply connected Riemann surface is conformally equivalent to exactly one of $\mathbb{D}$, $\mathbb{C}$ or $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.*

Every Riemann surface has exactly one of $\mathbb{D}$, $\mathbb{C}$, $\widehat{\mathbb{C}}$ as its universal covering surface. So given a Riemann surface M and its universal cover S, which is either $\mathbb{D}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$, and let G be the covering group of S over M, then M is conformally equivalent to S/G. Any Riemann surface which has $\mathbb{D}$ as its universal cover is called hyperbolic. Any non-trivial covering group of $\mathbb{C}$ is generated by either one or two translations, corresponding respectively to an infinite cylinder or a torus, and any Riemann surface with $\widehat{\mathbb{C}}$ as its universal covering surface is exactly $\widehat{\mathbb{C}}$, and so most Riemann surfaces are hyperbolic.

Exercise 1.5.1. Let M be a closed hyperbolic Riemann surface. The automorphism group of M, $\mathrm{Aut}(M)$, is the group of conformal self-mappings of M. Show that $\mathrm{Aut}(M)$ is a finite group.

In fact, Hurwitz proved in [26] that

$$|\mathrm{Aut}(M)| \leq 84(g-1),$$

where g is the genus of M.