# PART I

# THE LINEAR CAUCHY PROBLEM

# 1

# LINEAR CAUCHY PROBLEM WITH CONSTANT COEFFICIENTS

*The general Cauchy problem*

Let $d \geq 1$ be the space dimension and $x = (x_1, \dots, x_d)$ denote the space variable, $t$ being the time variable. The Cauchy problem that we consider in this section is posed in the whole space $\mathbb{R}^d$, while $t$ ranges on an interval, typically $(0, T)$, where $T \leq +\infty$.

A constant-coefficient first-order system is determined by $d+1$ matrices $A^1, \dots, A^d, B$ given in $\mathbf{M}_n(\mathbb{R})$, where $n \geq 1$ is the *size* of the system. Then the Cauchy problem consists in finding solutions $u(x,t)$ of

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha \frac{\partial u}{\partial x_\alpha} = Bu + f, \tag{1.0.1}$$

where $f = f(x,t)$ and the initial datum $u(\cdot, t = 0) = a$ are given in suitable functional spaces. To shorten the notation, we shall rewrite equivalently

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = Bu + f.$$

When $f \equiv 0$, the Cauchy problem is said to be *homogeneous*. A well-posedness property holds for the homogeneous problem when, given $a$ in a functional space $X$, there exists one and only one solution $u$ in $\mathscr{C}(0, T; Y)$, for some other functional space $Y$, the map

$$\begin{aligned} X &\to \mathscr{C}(0, T; Y) \\ a &\mapsto u \end{aligned}$$

being continuous. 'Solution' is understood here in the distributional sense. Existence and continuity imply $X \subset Y$, since the map $a \mapsto u(0)$ must be continuous. We use the general notation

$$\begin{aligned} X &\stackrel{S_t}{\to} Y \\ a &\mapsto u(t). \end{aligned}$$

Since a homogeneous system is, at a formal level, an autonomous differential equation with respect to time, we should like to have the semigroup property

$$S_{t+s} = S_t \circ S_s, \quad s, t \geq 0,$$

this of course requires that $Y = X$. We then say that the homogeneous Cauchy problem defines a continuous semigroup if for every initial data $a \in X$, there exists a unique distributional solution of class $\mathscr{C}(\mathbb{R}^+; X)$. Note that the word 'continuous' relies on the continuity with respect to time of the solution, but not on the continuity of $t \mapsto S_t$ in the operator norm. Semigroup theory actually tells us that, if $X$ is a Banach space, the continuity in the operator norm corresponds to ordinary differential equations, a context that does not apply in PDEs.

When the homogeneous Cauchy problem defines a continuous semigroup on a functional space $X$, we expect to solve the non-homogeneous one using *Duhamel's formula*:

$$u(t) = S_t a + \int_0^t S_{t-s} f(s) \mathrm{d}s, \tag{1.0.2}$$

provided that at least $f \in L^1(0, T; X)$. For this reason, we focus on the homogeneous Cauchy problem and content ourselves in constructing the semigroup.

Before entering into the theory, let us remark that, since (1.0.1) writes

$$\frac{\partial u}{\partial t} = Pu + f,$$

where $P$ is a differential operator of order less than or equal to one, the order with respect to time of this evolution equation, the Cauchy–Kowalevska theory applies. For instance, if $f = 0$, analytic initial data yield unique analytic solutions. However, these solutions exist only on a short time interval $(0, T^*(a))$. Since analytic data are unlikely in real life, and since local solutions are of little interest, we shall not concern ourselves with this result.

## 1.1 Very weak well-posedness

We first look at the necessary conditions for a very weak notion of well-posedness, where $X = \mathscr{S}(\mathbb{R}^d)$ (the Schwartz class) and $Y = \mathscr{S}'(\mathbb{R}^d)$, the set of tempered distributions. Surprisingly, this analysis will provide us with a rather strong necessary condition, sometimes called *weak hyperbolicity*[1].

Let us assume that the homogeneous Cauchy problem is well-posed in this context. Let $a$ be a datum and $u$ be the solution. From the equation

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha \frac{\partial u}{\partial x_\alpha} = Bu, \tag{1.1.3}$$

[1]Some authors call it simply *hyperbolicity*, and use the term *strong hyperbolicity* for the notion that we shall call *hyperbolicity*. Thus, depending on the authors, there is either the weak and normal hyperbolicities, or the normal and strong ones.

we obtain $u \in \mathscr{C}^\infty(0,T;Y)$. This allows us to Fourier transform (1.1.3) in the spatial directions. We obtain that (1.1.3) is equivalent to

$$\frac{\partial \hat{u}}{\partial t} + i \sum_{\alpha=1}^{d} \eta_\alpha A^\alpha \hat{u} = B\hat{u}.$$

Using the notation

$$A(\eta) := \sum_{\alpha=1}^{d} \eta_\alpha A^\alpha,$$

we rewrite this equation as an ODE in $t$, parametrized by $\eta$

$$\frac{\partial \hat{u}}{\partial t} = (B - iA(\eta))\hat{u}. \tag{1.1.4}$$

Since $\hat{u}(\cdot,0) = \hat{a}$, the solution of (1.1.4) is explicitly given by

$$\hat{u}(\eta,t) = \mathrm{e}^{t(B-iA(\eta))}\hat{a}(\eta). \tag{1.1.5}$$

By well-posedness (1.1.5) defines a tempered distribution for every choice of $\hat{a}$ in the Schwartz class, continuously in time. In other words, the bilinear map

$$(\phi,\psi) \mapsto \int_{\mathbb{R}^d} \psi(\eta)^* \mathrm{e}^{t(B-iA(\eta))}\phi(\eta)\,\mathrm{d}\eta, \tag{1.1.6}$$

which is well-defined for compactly supported smooth vector fields $\phi$ and $\psi$, is continuous in the Schwartz topology, uniformly for $t$ in compact intervals.

Let $\lambda$ be a simple eigenvalue of $A(\xi)$ for some $\xi \in \mathbb{R}^d$. Then, there is a $\mathscr{C}^\infty$ map $(t,\sigma) \mapsto (\mu, r)$, defined on a neighbourhood $\mathcal{W}$ of $(0,\xi)$, such that $\mu(0,\xi) = -i\lambda$ and

$$(t^2 B - iA(\sigma))r(t,\sigma) = \mu(t,\sigma)r(t,\sigma), \quad \|r\| \equiv 1.$$

Let us choose a non-zero compactly supported smooth function $\theta : \mathbb{R}^d \to \mathbb{C}$ with $\theta(0) \neq 0$. Then, for small enough $t > 0$, the condition $\eta - t^{-2}\xi \in \operatorname{Supp}\theta$ implies $(t, t^2\eta) \in \mathcal{W}$. For such a $t$, we may define two compactly supported smooth vector fields by

$$\phi^t(\eta) := \theta(\eta - t^{-2}\xi)r(t,t^2\eta), \quad \psi^t(\eta) := \theta(\eta - t^{-2}\xi)\ell(t,t^2\eta),$$

where $\ell$ is an eigenfield of the adjoint matrix $(t^2B - iA(\sigma))^*$, defined and normalized as above. We then apply (1.1.6) to $(\phi^t, \psi^t)$. The sequence $(\phi^t)_{t\to 0}$ is bounded in the Schwartz topology, and similarly is $(\psi^t)_{t\to 0}$. Therefore

$$\int_{\mathbb{R}^d} (\psi^t)^* \mathrm{e}^{t(B-\mathrm{i}A(\eta))}\phi^t\,\mathrm{d}\eta = \int_{\mathbb{R}^d} \mathrm{e}^{\mu(t,t^2\eta)/t}(\ell\cdot r)(t,t^2\eta)|\theta(\eta - \xi/t^2)|^2\mathrm{d}\eta$$

is bounded as $t \to 0$. Since it behaves like $c\exp(-\mathrm{i}\lambda/t)$ for a non-zero constant $c$, we conclude that $\operatorname{Im}\lambda \leq 0$. Applying also this conclusion to the simple eigenvalue $\bar{\lambda}$, we find that $\lambda$ is real.

The case of an eigenvalue of constant multiplicity in some open set of frequencies $\eta$ can be treated along the same ideas; it must be real too. Finally, the points $\eta$ at which the multiplicities are not locally constant form an algebraic submanifold, thus a set of void interior. By continuity, the reality must hold everywhere. We have thus proved

**Proposition 1.1** *The* $(\mathscr{S}, \mathscr{S}')$ *well-posedness requires that the spectrum of* $A(\xi)$ *be real for all* $\xi$ *in* $\mathbb{R}^d$.

When $(\mathscr{S}, \mathscr{S}')$ well-posedness does not hold, a Hadamard instability occurs: for most (in the Baire sense) data $a$ in $\mathscr{S}$, and for all $T > 0$, the Cauchy problem does not admit any solution of class $\mathscr{C}(0, T; \mathscr{S}')$. This is a consequence of the Principle of Uniform Boundedness.

**Example** The Cauchy–Riemann equations provide the simplest system for which this instability holds. One has $d = 1$, $n = 2$:

$$\partial_t u_1 + \partial_x u_2 = 0, \quad \partial_t u_2 - \partial_x u_1 = 0.$$

This example shows that a boundary value problem for a system of partial differential equations may be well-posed though the corresponding Cauchy problem is ill-posed.

The converse of Proposition 1.1 does not hold in general, mainly because of the interaction between non-semisimple eigenvalues of $A(\xi)$ with the mixing induced by $B$. Let us take again a simple example with $d = 1$, $n = 2$, and

$$A = A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since the matrix

$$\exp(-i\xi A) = I_2 - i\xi A$$

has polynomial growth, the Cauchy problem for the operator $\partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is well-posed in the $(\mathscr{S}, \mathscr{S}')$ sense, and even in the $(\mathscr{S}, \mathscr{S})$ sense. Actually, its solution is explicitly given by

$$u_1(t) = a_1 - t a_2', \quad u_2(t) \equiv a_2.$$

(We see that there is an immediate loss of regularity.) However, with our non-zero $B$, the matrix $M := t(B - i\xi A)$ satisfies $M^2 = -it^2 \xi I_2$, which implies that

$$\exp(t(B - i\xi A)) = \cos\omega \, I_2 + \frac{\sin\omega}{\omega} M,$$

where $\omega = t(i\xi)^{1/2}$. Since

$$\operatorname{Im} \omega = \pm t \left| \frac{\xi}{2} \right|^{1/2},$$

we see that offdiagonal coefficients of $\exp M$ grow like $\exp(c|\xi|^{1/2})$ as $\xi$ tends to infinity, provided $t \neq 0$. Then a calculation similar to the one in the proof of Proposition 1.1 shows that this Cauchy problem is ill-posed in the $(\mathscr{S}, \mathscr{S}')$ sense.

## 1.2 Strong well-posedness

The previous example suggests that the notion of well-posedness in the (rather weak) $(\mathscr{S}, \mathscr{S}')$ sense might not be stable under small disturbance (the instability result would be the same with $\epsilon B$ instead of $B$). For this reason, we shall merely consider the well-posedness when $Y = X$ and $X$ is a Banach space. We then speak about strong well-posedness in $X$ (or $X$-well-posedness). When this holds, the map $S_t : a \mapsto u(t)$ defines a continuous semigroup on $X$. It can be shown that if $X$ is a Banach space, there exist two constants $c$, $\omega$, such that

$$\|S_t\|_{\mathcal{L}(X)} \leq c\mathrm{e}^{\omega t}, \tag{1.2.7}$$

**Proposition 1.2** *Let $X$ be a Banach space. Then well-posedness (with $Y = X$) for some $B \in \mathbf{M}_n(\mathbb{R})$ implies the same property for all $B$.*

This amounts to saying that well-posedness is a property of $(A^1, \dots, A^d)$ alone.

**Proof** Assume strong well-posedness for a given matrix $B_0$. The problem

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha \frac{\partial u}{\partial x_\alpha} = B_0 u \tag{1.2.8}$$

defines a continuous semigroup $(S_t)_{t\geq 0}$. One has (1.2.7) with suitable constants $c$ and $\omega$. From Duhamel's formula, (1.1.3) with a matrix $B = B_0 + C$ instead of $B_0$, is equivalent to

$$u(t) = S_t a + \int_0^t S_{t-s} C u(s) \mathrm{d}s. \tag{1.2.9}$$

Then we can solve (1.2.9) by a Picard iteration. Let us denote by $Ru$ the right-hand side of (1.2.9), and $I = (0, T)$ (with $T > 0$) a time interval where we look for a solution. Because of (1.2.7), there exists a large enough $N$ so that $R^N$ is contractant on $\mathscr{C}(I; X)$. Therefore, there exists a unique solution of (1.1.3) in $\mathscr{C}(I; X)$. Since $T$ is arbitrary, the solution is global in time. □

### 1.2.1 *Hyperbolicity*

We first consider spaces $X$ where the Fourier transform defines an isomorphism onto some other Banach space $Z$. Typically, $X$ will be a Sobolev space $H^s(\mathbb{R}^d)^n$ and $Z$ is a weighted $L^2$-space:

$$Z = L^2_s(\mathbb{R}^d)^n, \quad L^2_s(\mathbb{R}^d) := \{v \in L^2_{\mathrm{loc}}(\mathbb{R}^d)\, ;\, (1 + |\xi|^2)^{s/2} v \in L^2(\mathbb{R}^d)\}.$$

Because of this example, we shall assume that multiplication by a measurable function $g$ defines a continuous operator from $Z$ to itself if and only if $g$ is bounded.

Looking for a solution $u \in \mathscr{C}(I; X)$ of (1.1.3) is simply looking for a solution $v \in \mathscr{C}(I; Z)$ of

$$\frac{\partial v}{\partial t} = (B - iA(\eta))v, \quad v(\eta, 0) = \hat{a}(\eta). \tag{1.2.10}$$

Thanks to Proposition 1.2, we may restrict ourselves to the case where $B = 0_n$. Then $v$ must obey the formula

$$v(\eta, t) = \mathrm{e}^{-\mathrm{i}tA(\eta)}\hat{a}(\eta),$$

where $\hat{a}$ is given in $Z$. In order that $v(t)$ belong to $Z$ for all $\hat{a}$, it is necessary and sufficient that $\eta \mapsto \exp(-itA(\eta))$ be bounded. Since $tA(\eta) = A(t\eta)$, this is equivalent to writing

$$\sup_{\xi \in \mathbb{R}^d} \| \exp(iA(\xi))\| < +\infty. \tag{1.2.11}$$

Let us emphasize that this property does not depend on the time $t$, once $t \neq 0$.

**Definition 1.1** *A first-order operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

*is called* hyperbolic *if the corresponding symbol $\xi \mapsto A(\xi)$ satisfies (1.2.11).*

*More generally, a system (1.0.1) (whatever $B$ is) that satisfies (1.2.11) is called a* hyperbolic[2] *system of first-order PDEs.*

After having proven that hyperbolicity is a necessary condition, we show that it is sufficient for the $H^s$-well-posedness. It remains to prove the continuity of $t \mapsto v(t)$ with values in $Z$, when $\hat{a}$ is given in $Z$. For that, we write

$$\|v(\tau) - v(t)\|_Z^2 = \int_{\mathbb{R}^d} \left| \left(\mathrm{e}^{-\mathrm{i}\tau A(\eta)} - \mathrm{e}^{-\mathrm{i}tA(\eta)}\right) \hat{a}(\eta)\right|^2 (1 + |\eta|^2)^s \mathrm{d}\eta.$$

Thanks to (1.2.11), the integrand is bounded by $c|\hat{a}(\eta)|^2(1 + |\eta|^2)^s$, an integrable function, independent of $\tau$. Likewise, it tends pointwisely to zero, as $\tau \to t$. Lebesgue's Theorem then implies that

$$\lim_{\tau \to t} \|v(\tau) - v(t)\|_Z = 0.$$

Let us summarize the results that we obtained:

[2]Some authors write *strongly hyperbolic* in this definition and keep the terminology *hyperbolic* for those systems that are well-posed in $\mathscr{C}^\infty$, that is whose a priori estimates may display a loss of derivatives.

**Theorem 1.1**

- *Let $s$ be a real number. The Cauchy problem for*
$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = 0 \qquad (1.2.12)$$
*is $H^s$-well-posed if and only if this system is hyperbolic.*
- *If the operator $L$ (defined as above) is hyperbolic, then the Cauchy problem for (1.1.3) is $H^s$-well-posed for every real number $s$.*
- *In particular, the Cauchy problem is well-posed in $H^s$ if and only if it is well-posed in $L^2$.*

Let us point out that hyperbolicity does not involve the matrix $B$.

Since the well-posedness in a Hilbertian Sobolev space holds or does not, independently of the regularity level $s$, we feel free to rename this property *strong well-posedness.*

**Backward Cauchy problem** We considered up to now the forward Cauchy problem, namely the determination of $u(t)$ for times $t$ larger than the initial time. Its well-posedness within $L^2$ was shown to be equivalent to hyperbolicity. Reversing the time arrow amounts to making the change $\partial_t \mapsto -\partial_t$. This has the same effect as changing the matrices $A^\alpha$ into $-A^\alpha$. The $L^2$-well-posedness of the Cauchy problem is thus equivalent to the hyperbolicity of the new system
$$\partial_s u - \sum_\alpha A^\alpha \partial_\alpha u = -Bu.$$
This writes as
$$\sup_{\xi\in\mathbb{R}^d} \| \exp(-iA(\xi))\| < +\infty,$$
which is the same as (1.2.11), *via* the change of dummy variable $\xi \mapsto -\xi$. Finally, the strong well-posedness of backward and forward Cauchy problems are equivalent to each other. For a hyperbolic system and a data $a \in H^s(\mathbb{R}^d)^n$, there exists a unique solution of (1.1.3) $u \in \mathscr{C}(\mathbb{R}; H^s(\mathbb{R}^d)^n)$ such that $u(0) = a$. Let us emphasize that here, $t$ ranges on the whole line, not only on $\mathbb{R}^+$.

### 1.2.2 *Distributional solutions*

When (1.1.3) is hyperbolic, one can also solve the Cauchy problem for data in the set $\mathscr{S}'$ of tempered distribution. For that, we again use the Fourier transform since it is an automorphism of $\mathscr{S}'$. We again define $\hat{u}$ by the formula (1.1.5). We only have to show that this definition makes sense in $\mathscr{S}'$ for every $t$, and that $u$ is continuous from $\mathbb{R}_t$ to $\mathscr{S}'$. For that, we have to show that $X(t) := \exp(t(B - iA(\eta)))$ is a $\mathscr{C}^\infty$ function of $\eta$, with slow growth at infinity, locally uniformly in time. We shall show that its derivatives are actually bounded with respect to $\eta$. The regularity is trivial, and we already know that $X(t)$ is bounded in $\eta$, locally in time. Denoting by $X_\alpha$ the derivative with respect to $\eta_\alpha$, we have

$$\frac{\mathrm{d}X_\alpha}{\mathrm{d}t} = (B - iA(\eta))X_\alpha - iA^\alpha X,$$

and therefore

$$\frac{\mathrm{d}(X^{-1}X_\alpha)}{\mathrm{d}t} = -iX^{-1}A^\alpha X.$$

Using Duhamel's formula, as in the proof of Proposition 1.2, we see that

$$\|X(t)\| \le c(1 + \|B\|\,|t|),$$

from which we deduce

$$\|X^{-1}X_\alpha\| \le \frac{(1 + \|B\|\,|t|)^3 - 1}{3\|B\|} c^2 \|A^\alpha\|.$$

Finally, we obtain

$$\|X_\alpha\| \le \frac{(1 + \|B\|\,|t|)^4}{3\|B\|} c^3 \|A^\alpha\|.$$

We leave the reader to estimate the higher derivatives and complete the proof of the following statement. The case of data in the Schwartz class is done in exactly the same way, since the Fourier transform is an automorphism of $\mathscr{S}$ and that $\mathscr{S}$ is stable under multplication by $\mathscr{C}^\infty$ functions with slow growth.

**Proposition 1.3** *If $L$ is hyperbolic, then the Cauchy problem for (1.1.3) is well-posed in both $\mathscr{S}$ and $\mathscr{S}'$.*

### 1.2.3 *The Kreiss' matrix Theorem*

Of course, since $L^2$-well-posedness implies $(\mathscr{S}, \mathscr{S}')$-well-posedness, hyperbolicity implies that the spectrum of $A(\xi)$ is real for all $\xi$ in $\mathbb{R}^d$. It implies even more, that all $A(\xi)$ are diagonalizable. Though these two facts have a rather simple proof here, they do not characterize completely hyperbolic systems. We shall therefore describe the characterization obtained by Kreiss [102, 104]. This is an application of a deeper result that deals with strong well-posedness of general constant-coefficient evolution problems. However, since we focus only on first-order systems, we content ourselves with a statement with a simpler proof, due to Strang [199].

**Theorem 1.2** *Let $\xi \mapsto A(\xi)$ be a linear map from $\mathbb{R}^d$ to $\mathbf{M}_n(\mathbb{C})$. Then the following properties are equivalent to each other:*

*i) Every $A(\xi)$ is diagonalizable with pure imaginary eigenvalues, uniformly with respect to $\xi$:*

$$A(\xi) = P(\xi)^{-1} \mathrm{diag}(i\rho_1, \dots, i\rho_n) P(\xi), \quad (\rho_1(\xi), \dots, \rho_n(\xi) \in \mathbb{R}),$$

*with*

$$\|P(\xi)^{-1}\| \cdot \|P(\xi)\| \leq C', \quad \forall \xi \in \mathbb{R}^d. \tag{1.2.13}$$

*ii) There exists a constant $C > 0$, such that*

$$\left\| \mathrm{e}^{tA(\xi)} \right\| \leq C, \quad \forall \xi \in \mathbb{R}^d, \forall t \geq 0. \tag{1.2.14}$$

*iii) There exists a constant $C > 0$, such that*

$$\left\| (zI_n - A(\xi))^{-1} \right\| \leq \frac{C}{\mathrm{Re}\ z}, \quad \forall \xi \in \mathbb{R}^d, \forall \mathrm{Re}\ z > 0. \tag{1.2.15}$$

Note that, replacing $(z, \xi)$ by $(-z, -\xi)$, we also obtain (1.2.15) with Re $z \neq 0$. Applying Theorem 1.2, we readily obtain the following.

**Theorem 1.3** *The Cauchy problem for a first-order system*

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = 0, \quad x \in \mathbb{R}^d$$

*is $H^s$-well-posed if and only if the following two properties hold.*

- *The matrices $A(\xi)$ are diagonalizable with real eigenvalues,*

$$A(\xi) = P(\xi)^{-1} \mathrm{diag}(\rho_1(\xi), \dots, \rho_n(\xi)) P(\xi), \quad (\rho_1, \dots, \rho_n \in \mathbb{R}).$$

- *Their diagonalization is well-conditioned (one may also say that the matrices $A(\xi)$ are* uniformly *diagonalizable) : $\sup_{\xi \in S^{d-1}} \|P(\xi)^{-1}\| \cdot \|P(\xi)\| < +\infty$.*

**Proof** The fact that *i)* implies *ii)* is proved easily. Actually,

$$\|\mathrm{e}^{tA(\xi)}\| = \|P^{-1} \mathrm{e}^{tD} P\| \leq C' \|\mathrm{e}^{tD}\|.$$

When $D$ is diagonal with pure imaginary entries, $\exp(tD)$ is unitary, and the right-hand side equals $C'$.

The fact that *ii)* implies *iii)* is easy too. The following equality holds provided the integral involved in it converges in norm

$$(A - zI_n) \int_0^\infty \mathrm{e}^{-zt} \mathrm{e}^{tA} \mathrm{d}t = -I_n. \tag{1.2.16}$$

Because of (1.2.14), the integral converges for every $z \in \mathbb{C}$ with positive real part. This gives a bound for the inverse of $zI_n - A$, of the form

$$\|(zI_n - A)^{-1}\| \leq \frac{C}{\mathrm{Re}\ z}, \quad \mathrm{Re}\ z > 0.$$

It remains to prove that *iii)* implies *i)*. Thus, let us assume (1.2.15). Replacing $(z, \xi)$ by $(-z, -\xi)$, we see that the bound holds for Re $z \neq 0$, with $|\mathrm{Re}\ z|$ in the denominator. Thus the spectrum of $A(\xi)$ is purely imaginary.

Actually, $A(\xi)$ is diagonalizable, for if there were a non-trivial Jordan part, then $(zI_n - A(\xi))^{-1}$ would have a pole of order two or more, contradicting (1.2.15). Therefore, $A(\xi)$ admits a spectral decomposition

$$A(\xi) = i \sum_j \rho_j E_j,$$

where $\rho_j$ is real and $E_j = E_j(\xi)$ is a projector ($E_j^2 = E_j$), with

$$E_j E_k = 0_n, \quad (k \neq j), \qquad \sum_j E_j = I_n.$$

Let us define

$$H = H(\xi) := \sum_j E_j E_j^*,$$

which is a positive-definite Hermitian matrix. Since $A(\xi)^* = -\sum_j \rho_j E_j^*$, it holds that

$$H(\xi)A(\xi) = -A(\xi)^* H(\xi),$$

from which it follows that $H(\xi)^{1/2}A(\xi)H(\xi)^{-1/2}$ is skew-Hermitian. As such, it is diagonalizable through a unitary transformation. Therefore $A(\xi) = P(\xi)^{-1}D(\xi)P(\xi)$, where $D(\xi)$ is diagonal with pure imaginary eigenvalues, and $P(\xi) = U(\xi)H^{1/2}$, where $U(\xi)$ is a unitary matrix.

We finish by proving that $P(\xi)$ is uniformly conditioned. Since $\|P^{\pm 1}\| = \|H^{\pm 1/2}\| = \|H\|^{\pm 1/2}$, this amounts to proving that $\|H\| \cdot \|H^{-1}\|$ is uniformly bounded. On the one hand, it holds that

$$|v|^2 = \left| \sum_j E_j v \right|^2 \leq n \sum_j |E_j v|^2 = n |H^{1/2} v|^2,$$

so that $\|H^{-1/2}\| \leq \sqrt{n}$. On the other hand, applying (1.2.15) to $\epsilon + i\rho_k$, we have

$$\| \sum_j (\epsilon + i\rho_k - i\rho_j)^{-1} E_j \| \leq \frac{C}{|\epsilon|}.$$

Letting $\epsilon \to 0$, we deduce that $\|E_j\| \leq C$, independently of $\xi$. It follows that $\|H\| \leq nC^2$. □

**Remarks**

*i)* A more explicit characterization of hyperbolic symbols has been established by Mencherini and Spagnolo when $n = 2$ or $n = 3$; see [129].

*ii)* The following example ($n = 3$ and $d = 2$), known as *Petrowski's example*, shows that the well-conditioning can fail for systems in which all matrices

$A(\xi)$ are diagonalizable with real eigenvalues. Let us take

$$A^1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One checks easily that the eigenvalues of $A(\xi)$ are real and distinct for $\xi_1 \neq 0$, while $A^2$ is already diagonal. Hence, $A(\xi)$ is always diagonalizable over $\mathbb{R}$. However, as $\xi_1$ tends to zero, one eigenvalue is identically zero, associated to the eigenvector $(\xi_2, \xi_1, -\xi_1)^T$, while another one is small, $\lambda \sim -\xi_1^2\xi_2^{-1}$, associated to $(\xi_2, 0, \lambda\xi_2\xi_1^{-1})^T$. Both eigenvectors have the same limit $(\xi_2, 0, 0)^T$, which shows that $P(\xi)$ is unbounded as $\xi_1$ tends to zero. See a similar example in [108]. Oshime [155] has shown that Petrowsky's example is somehow canonical when $d = 3$. On the other hand, Strang [199] showed that when $n = 2$, the diagonalizability of every $A(\xi)$ is equivalent to hyperbolicity, and that such operators are actually Friedrichs symmetrizable in the sense of the next section.

*iii)* Uniform diagonalizability of $A(\xi)$ within real matrices has been shown by Kasahara and Yamaguti [93, 221] to be necessary and sufficient in order that the Cauchy problem for

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = Bu$$

be $\mathcal{C}^\infty$-well-posed for every matrix $B \in \mathbf{M}_n(\mathbb{R})$. Of course, the sufficiency follows from Theorem 1.3 and Proposition 1.2. The necessity statement is even stronger than the one suggested by the example given in Section 1.1, since the diagonalizability within $\mathbb{R}$ is not sufficient. For instance, if $A(\xi)$ is given as in the Petrowski example, there are matrices $B$ for which the Cauchy problem is ill-posed in the Hadamard sense.

### 1.2.4 *Two important classes of hyperbolic systems*

We now distinguish two important classes of hyperbolic systems.

**Definition 1.2** *An operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

*is said to be* symmetric in Friedrichs' sense *[63], or simply* Friedrichs symmetric, *if all matrices $A^\alpha$ are symmetric; one may also say* symmetric hyperbolic. *More generally, it is* Friedrichs symmetrizable *if there exists a symmetric positive-definite matrix $S_0$ such that every $S_0 A^\alpha$ is symmetric.*

*An operator $M$ as above is said to be* constantly[3] hyperbolic *if the matrices $A(\xi)$ are diagonalizable with real eigenvalues and, moreover, as $\xi$ ranges along*

[3]We employ this shortcut in lieu of *hyperbolic with characteristic fields of constant multiplicities.*

$S^{d-1}$*, the multiplicities of eigenvalues remain constant. In the special case where all eigenvalues are real and simple for every* $\xi \in S^{d-1}$*, we say that the operator is* strictly hyperbolic.

Let us point out that in a constantly hyperbolic operator, the eigenvalues may have non-equal multiplicities, but the set of multiplicities remains constant as $\xi$ varies. This implies in particular that the eigenspaces depend analytically on $\xi$ for $\xi \neq 0$. This fact easily follows from the construction of eigenprojectors as Cauchy integrals (see the section 'Notations'.) To a large extent, the theory of constantly hyperbolic systems does not differ from the one of strictly hyperbolic systems. But the analysis is technically simpler in the latter case. This is why the theory of strictly hyperbolic operators was developed much further in the first few decades.

**Theorem 1.4** *If an operator is Friedrichs symmetrizable, or if it is constantly hyperbolic, then it is hyperbolic.*

**Proof** Let the operator be Friedrichs symmetrizable by $S_0$. Then $S_0^{-1}$ is positive-definite and admits a (unique) square root $R$ symmetric positive-definite (see [187], page 78). Let us denote $S_0 A^\alpha$ by $S^\alpha$, and $S(\xi) = \sum_\alpha \xi_\alpha S^\alpha$ as usual. Then

$$A(\xi) = S_0^{-1} S(\xi) = R(RS(\xi)R)R^{-1}.$$

The matrix $RS(\xi)R$ is real symmetric and thus may be written as $Q(\xi)^T D(\xi) Q(\xi)$, where $Q$ is orthogonal. Then $A(\xi)$ is conjugated to $D(\xi)$, $A(\xi) = P(\xi)^{-1} D(\xi) P(\xi)$, with $P(\xi) = Q(\xi) R^{-1}$ and $P(\xi)^{-1} = RQ(\xi)^T$. Since our matrix norm is invariant under left or right multiplication by unitary matrices, we have

$$\|P(\xi)\| \, \|P(\xi)^{-1}\| = \|R\| \, \|R^{-1}\| = \sqrt{\rho(S_0)\rho(S_0^{-1})},$$

a number independent of $\xi$. The diagonalization is thus well-conditioned.

Let us instead assume that the system is constantly hyperbolic. The eigenspaces are continuous functions of $\xi$ in $S^{d-1}$. Choosing continuously a basis of each eigenspace, we find locally an eigenbasis of $A(\xi)$, which depends continuously on $\xi$. This amounts to saying that, along every contractible subset of $S^{d-1}$, the matrices $A(\xi)$ may be diagonalized by a matrix $P(\xi)$, which depends continuously on $\xi$. If the set is, moreover, compact (for instance, a half-sphere), we obtain that $A(\xi)$ is diagonalizable with a uniformly bounded condition number. We now cover the sphere by two half-spheres and obtain a diagonalization of $A(\xi)$ that is well-conditioned on $S^{d-1}$ (though possibly not continuously diagonalizable on the sphere). $\square$

In the following example, though a symmetric as well as a strictly hyperbolic one, the diagonalization of the matrices $A(\xi)$ cannot be done continuously for all

$\xi \in S^1$ :

$$\partial_t u + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 u + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 u = 0. \tag{1.2.17}$$

Here, $\mathrm{Sp}(A(\xi)) = \{-|\xi|, |\xi|\}$. Each eigenvector, when followed continuously as $\xi$ varies along $S^1$, rotates with a speed half of the speed of $\xi$. For $\xi = (\cos\theta, \sin\theta)^T$ and $\theta \in [0, 2\pi)$, the eigenvectors are

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \quad \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

The eigenbasis is reversed after one loop around the origin. This shows that the matrix $P(\xi)$ cannot be chosen continuously. In other words, the eigenbundle is non-trivial.

### 1.2.5 *The adjoint operator*

Let $L$ be a hyperbolic operator as above. We define as usual the *adjoint operator* $L^*$ by the identity

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} (v \cdot (Lu) - u \cdot (L^* v)) \mathrm{d}x \, \mathrm{d}t = 0, \tag{1.2.18}$$

for every $u, v \in \mathscr{D}(\mathbb{R}^{d+1})^n$. Notice that the scalar product under consideration is the one in the $L^2$-space in $(x,t)$-variables.

With $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$, an integration by parts gives immediately the formula

$$L^* = -\partial_t - \sum^{\alpha} (A^\alpha)^T \partial_\alpha.$$

The matrix $A(\xi)^T$, being similar to $A(\xi)$, is diagonalizable. Since $A(\xi)^T$ is diagonalized by $P(\xi)^{-T}$ (with the notations of Theorem 1.3), and since the matrix norm is invariant under transposition, we see that $-L^*$ is hyperbolic too. If $L$ is strictly, or constantly, hyperbolic, so is $L^*$. If $L$ is Friedrichs symmetrizable, with $S^0 \in \mathbf{SDP}_n$ and $S^\alpha := S^0 A^\alpha \in \mathbf{Sym}_n$, then $(S^0)^{-1}$ symmetrizes $-L^*$ since it is positive-definite and $(S^0)^{-1} (A^\alpha)^T = (S^0)^{-1} S^\alpha (S^0)^{-1}$ is symmetric. Therefore, $L^*$ is Friedrichs symmetrizable.

The adjoint operator will be used in the existence theory of the Cauchy problem (the duality method) or in the uniqueness theory (Holmgren's argument), the latter being useful even in the quasilinear case. Both aspects are displayed in Chapter 2.

### 1.2.6 *Classical solutions*

Let the system (1.1.3) be hyperbolic. According to Theorem 1.1, the Cauchy problem is well-posed in $H^s$. Using the system itself, we find that, whenever

$a \in H^s(\mathbb{R}^d)^n$,

$$u \in \mathscr{C}(\mathbb{R}; H^s(\mathbb{R}^d)^n) \cap \mathscr{C}^1(\mathbb{R}; H^{s-1}(\mathbb{R}^d)^n).$$

Let us assume that $s > 1 + d/2$. By Sobolev embedding, $H^s \subset \mathscr{C}^1$ and $H^{s-1} \subset \mathscr{C}$ hold. We conclude that all distributional first-order derivatives are actually continuous functions of space and time. Therefore, $u$ belongs to $\mathscr{C}^1(\mathbb{R}^d \times \mathbb{R})^n$ and is a classical solution of (1.1.3).

More generally, $a \in H^s(\mathbb{R}^d)^n$ with $s > k + d/2$ implies that $u$ is of class $\mathscr{C}^k$.

Let us consider the non-homogeneous Cauchy problem, with $a \in H^s(\mathbb{R}^d)^n$ and $f \in L^1(\mathbb{R}; H^s(\mathbb{R}^d)^n) \cap \mathscr{C}(\mathbb{R}; H^{s-1}(\mathbb{R}^d)^n)$ for $s > 1 + d/2$. Then Duhamel's formula immediately gives $u \in \mathscr{C}(\mathbb{R}; H^s(\mathbb{R}^d)^n)$, and the equation gives $\partial_t u \in \mathscr{C}(\mathbb{R}; H^{s-1}(\mathbb{R}^d)^n)$. We again conclude that $u$ is $\mathscr{C}^1$ and is a classical solution of (1.0.1).

Since $H^s(\mathbb{R}^d)^n$ is dense in normal functional spaces, as $L^2$ or $\mathscr{S}'$, we see that classical solutions are dense in weaker solutions, like those in $\mathscr{C}(\mathbb{R}; L^2(\mathbb{R}^d)^n)$. We shall make use of this observation each time when some identity trivially holds for classical solutions.

**The scalar case** When $n = 1$, the unknown $u(x, t)$ is scalar-valued and all matrices are real numbers, denoted by $a^1, \dots, a^n, b$. The supremum in (1.2.11) is equal to one, so that the equation is hyperbolic. It turns out that the Cauchy problem may be solved explicitly, thanks to the *method of characteristics*. Let $\vec{v}$ denote the vector with components $a^\alpha$. Then a classical solution of (1.1.3) satisfies, for all $y \in \mathbb{R}^d$,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(y + t\vec{v}, t) = b u(y + t\vec{v}, t),$$

which gives

$$u(y + t\vec{v}, t) = \mathrm{e}^{tb} a(y),$$

or

$$u(x, t) = \mathrm{e}^{tb} a(x - t\vec{v}). \tag{1.2.19}$$

This formula gives the distributional solution for $a \in \mathscr{S}'$ as well. The solution of the Cauchy problem for the non-homogeneous equation (1.0.1) is given by

$$u(x, t) = \mathrm{e}^{tb} a(x - t\vec{v}) + \int_0^t \mathrm{e}^{(t-s)b} f(x - (t - s)\vec{v}, s)\, \mathrm{d}s.$$

### 1.2.7 *Well-posedness in Lebesgue spaces*

The theory of the Cauchy problem is intimately related to Fourier analysis, which does not adapt correctly to Lebesgue spaces $L^p$ other than $L^2$. The procedure followed above requires that $\mathcal{F}$ be an isomorphism from some space $X$ to another one $Z$. It is known that $\mathcal{F}$ extends continuously from $L^p(\mathbb{R}^d)$ to its dual $L^{p'}(\mathbb{R}^d)$ when $1 \le p \le 2$, and only in these cases. Since $\mathcal{F}^{-1}$ is conjugated to $\mathcal{F}$ through complex conjugation, it satisfies the same property. Therefore, $\mathcal{F} : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ is not an isomorphism for $p < 2$, since $p' > 2$. From this remark, we cannot find a well-posedness result in $L^p$ for $p \neq 2$ by following the above strategy.

It has been proved actually by Brenner [22, 23] that, for hyperbolic systems, the Cauchy problem is ill-posed in $L^p$ for every $p \neq 2$, except in the case where the matrices $A^\alpha$ commute to each other. In this particular case, system (1.2.12) actually decouples into a list of scalar equations, for which (1.2.19) shows the well-posedness in every $L^p$. To see the decoupling, we recall that commuting matrices that are diagonalizable may be diagonalized in a common basis $\mathcal{B} = \{r_1, \ldots, r_n\}$ : $A^\alpha r_j = \lambda_j^\alpha r_j$. Let us decompose the unknown on the eigenbasis:

$$u(x,t) = \sum_1^n w_j(x,t) r_j.$$

Then each $w_j$ solves a scalar equation:

$$\partial_t w_j + \sum_\alpha \lambda_j^\alpha \partial_\alpha w_j = 0.$$

From the well-posedness of (1.2.12) and Duhamel's formula, we conclude that, for commuting matrices $A^\alpha$, the hyperbolic Cauchy problem for (1.1.3) is also well-posed in every $L^p$. The matrices $A^\alpha$ do not need to commute with $B$.

See Section 1.5.2 for an interpretation of the ill-posedness in $L^p$ ($p \neq 2$), in terms of dispersion and so-called Stricharz estimates.

## 1.3 Friedrichs-symmetrizable systems

A system in Friedrichs-symmetric form

$$S_0 \partial_t u + \sum_\alpha S^\alpha \partial_\alpha u = 0$$

may always be transformed into a symmetric system with $S_0 = I_n$, using the new unknown $\tilde{u} := S_0^{1/2} u$. For the rest of this section, we shall only consider symmetric systems of the form (1.1.3).

A symmetric system admits an additional conservation law[4] in the form

$$\partial_t |u|^2 + \sum_\alpha \partial_\alpha (A^\alpha u, u) = 0, \tag{1.3.20}$$

where $(\cdot,\cdot)$ denotes the canonical scalar product and $|u|^2 := (u,u)$. Equation (1.3.20) is satisfied at least for $\mathscr{C}^1$ solutions of the system, when[5] $B = 0$. It can be viewed as an *energy identity.* Since classical solutions are dense in $\mathscr{C}(\mathbb{R}; L^2(\mathbb{R}^d)^n)$,

[4] By *conservation law*, we mean an equality of the form $\mathrm{Div}_{x,t} \vec{F} = 0$ that derives formally from the equation or system under consideration.

[5] Otherwise, the right-hand side of (1.3.20) should be $2(Bu,u)$. In the non-homogeneous case, we add also $2(f,u)$.

and since

$$u \mapsto \partial_t |u|^2 + \sum_\alpha \partial_\alpha (A^\alpha u, u)$$

is a continuous map from this class into $\mathscr{D}'(\mathbb{R}^{d+1})$, we conclude that (1.3.20) holds whenever $a \in L^2(\mathbb{R}^d)^n$.

With suitable decay at infinity, (1.3.20) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |u(x,t)|^2 \mathrm{d}x = 0,$$

which readily gives

$$\|u(t)\|_{L^2} \equiv \|a\|_{L^2}. \tag{1.3.21}$$

Again, this identity is true for all data $a$ given in $L^2(\mathbb{R}^d)^n$, since

- it is trivially true for $a \in \mathscr{S}$, where we know that $u(t) \in \mathscr{S}$, since such functions decay fast at infinity,
- $\mathscr{S}$ is a dense subset of $L^2$.

### 1.3.1 *Dependence and influence cone*

Actually, we can do a better job from (1.3.20). Let us first consider classical solutions, for some matrix $B$. The set $\mathcal{V}$ of pairs $(\lambda, \nu)$ such that the symmetric matrix $\lambda I_n + A(\nu)$ is non-negative is a closed convex cone. Given a point $(X, T) \in \mathbb{R}^d \times \mathbb{R}$, we define a set $K$ by

$$K := \{(x,t)\,;\, \lambda(t-T) + (x-X)\cdot\nu \le 0, \forall (\lambda,\nu) \in \mathcal{V}\}.$$

As an intersection of half-spaces passing through $(X, T)$, $K$ is a convex cone with basis $(X, T)$, and its boundary $K$ has almost everywhere a tangent space, which is one of the hyperplanes $\lambda(t - T) + (x - X) \cdot \nu = 0$ for some $(\lambda, \nu)$ in the boundary of $\mathcal{V}$.

Given times $t_1 < t_2 < T$, we integrate the identity

$$\partial_t |u|^2 + \sum_\alpha \partial_\alpha (A^\alpha u, u) = 2(Bu, u)$$

on the truncated cone $K(t_1, t_2) := \{(x,t) \in K\,;\, t_1 < t < t_2\}$. Using Green's formula, we obtain

$$\int_{\partial K(t_1,t_2)} \left( n_0 |u|^2 + \sum_\alpha n_\alpha (A^\alpha u, u) \right) \mathrm{d}S = 2 \int_{K(t_1,t_2)} (Bu, u)\, \mathrm{d}x\, \mathrm{d}t, \tag{1.3.22}$$

where $\mathrm{d}S$ stands for the area element, while $\vec{n} = (n_1, \dots, n_d, n_0)$ is the outward unit normal. On the top $(t = t_2)$, $\vec{n} = (0, \dots, 0, 1)$, holds while on the bottom, $\vec{n} = (0, \dots, 0, -1)$. Denoting $\omega(t) := \{x\,;\,(x,t) \in K\}$, the corresponding contributions are thus

$$\int_{\omega(t_2)} |u(x,t_2)|^2 \mathrm{d}x - \int_{\omega(t_1)} |u(x,t_1)|^2 \mathrm{d}x.$$

On the lateral boundary, one has

$$\vec{n} = \frac{1}{\sqrt{\lambda^2 + |\nu|^2}}(\nu, \lambda)$$

for some $(\lambda, \nu)$ in $\mathcal{V}$, which depends on $(x,t)$. The parenthesis in (1.3.22) becomes

$$\frac{1}{\sqrt{\lambda^2 + |\nu|^2}}((\lambda I_n + A(\nu))u, u).$$

Thus the corresponding integral is non-negative. Denoting by $y(t)$ the integral of $|u(t)|^2$ over $\omega(t)$, it follows that

$$y(t_2) - y(t_1) \le 2 \int_{K(t_1,t_2)} (Bu, u)\, \mathrm{d}x\, \mathrm{d}t \le 2\|B\| \int_{t_1}^{t_2} y(t)\, \mathrm{d}t.$$

Then, from the Gronwall inequality, we obtain that

$$y(t_2) \le e^{2(t_2 - t_1)\|B\|} y(t_1).$$

In particular, for $0 < t < T$, we obtain

$$\int_{\omega(t)} |u(x,t)|^2 \mathrm{d}x \le e^{2t\|B\|} \int_{\omega(0)} |a(x)|^2 \mathrm{d}x. \tag{1.3.23}$$

Because of the density of classical solutions in the set of $L^2$-solutions, and since its terms are $L^2$-continuous, we find that (1.3.23) is valid for every $L^2$-solutions.

Inequality (1.3.23) contains the following fact: If $a$ vanishes identically on $\omega(0)$, then so does $u(t)$ on $\omega(t)$. Equivalently, the value of $u$ at the point $(X,T)$ (assuming that the solution is continuous) depends only on the restriction of the initial data $a$ to the set $\omega(0)$.

**Definition 1.3** *The set*

$$\omega(0) = \{x \in \mathbb{R}^d \,;\, (x - X)\cdot \nu \le \lambda T,\ \forall(\lambda, \nu) \in \mathcal{V}\}$$

*is the* domain of dependence *of the point* $(X,T)$.

Let us illustrate this notion with the system (1.2.17), to which we add a parameter $c$ having the dimension of a velocity:

$$\partial_t u + c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 u + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 u = 0.$$

Since

$$\lambda I_2 + A(\nu) = \begin{pmatrix} \lambda + c\nu_1 & c\nu_2 \\ c\nu_2 & \lambda - c\nu_1 \end{pmatrix},$$

the cone $\mathcal{V}$ is given by the inequality $c|\nu| \leq \lambda$. Thus the domain of dependence of $(X, T)$ is the ball centred at $X$ of radius $cT$.

We now fix a point $x$ at initial time and look at those points $(X, T)$ for which $x$ belongs to their domains of dependence. Let us define a convex cone $\mathcal{C}^+$ by

$$\mathcal{C}^+ := \{y \in \mathbb{R}^d \,;\, \lambda + y \cdot \nu \geq 0, \, \forall (\lambda, \nu) \in \mathcal{V}\}.$$

Defining $y = (X - x)/T$, we have $1 + y \cdot \nu \geq 0$, that is $y \in \mathcal{C}^+$. Therefore $X = x + Ty \in x + T\mathcal{C}^+$. We deduce that $u$ vanishes identically outside of $\operatorname{Supp} a + T\mathcal{C}^+$, where $a = u(\cdot, 0)$. We have thus proved a propagation property:

**Proposition 1.4** *Let the system (1.1.3) be symmetric. Given $a \in L^2(\mathbb{R}^d)^n$, let $u$ be the solution of the Cauchy problem. Then, for $t_1 < t_2$,*

$$\operatorname{Supp} u(t_2) \subset \mathit{Supp}\, u(t_1) + (t_2 - t_1)\mathcal{C}^+. \tag{1.3.24}$$

*Reversing the time arrow, we likewise have*

$$\mathit{Supp}\, u(t_1) \subset \mathit{Supp}\, u(t_2) + (t_2 - t_1)\mathcal{C}^-, \tag{1.3.25}$$

*where*

$$\mathcal{C}^- := \{y \in \mathbb{R}^d \,;\, \lambda + y \cdot \nu \geq 0, \, \forall \nu \in -\mathcal{V}\}.$$

This result naturally yields the notion:

**Definition 1.4** *Given a domain $\omega$ at initial time. The* influence domain *of $\omega$ at time $t > 0$ is the set $\omega + t\mathcal{C}^+$.*

**Remark** From Duhamel's formula, we extend the propagation property to the non-homogeneous problem. For instance, the solution for data $a \in L^2$ and $f \in L^1(0, T; L^2)$ satisfies

$$\operatorname{Supp} u(t) \subset (\operatorname{Supp} a + t\mathcal{C}^+) \cup \bigcup_{0<s<t} (\operatorname{Supp} f(s) + (t - s)\mathcal{C}^+). \tag{1.3.26}$$

### 1.3.2 *Non-decaying data*

Though the previous calculation applies only to solutions in $\mathscr{C}(\mathbb{R}; L^2)$, where we already know the uniqueness of a solution, it can be used to construct solutions for much more general data than square-integrable ones.

First, the inequality (1.3.23) implies a propagation with finite speed: if $a \in L^2(\mathbb{R}^d)^n$ and $t > 0$, the support of $u(t)$ is contained in the sum $\operatorname{Supp} a + t\mathcal{C}^+$. We now use the following facts:

- $L^2$ is dense in $\mathscr{S}'$,
- for $a$ in $\mathscr{S}'$, there exists a unique solution in $\mathscr{C}(\mathbb{R};\mathscr{S}')$ (see Proposition 1.3),
- the distributions that vanish on a given open subset of $\mathbb{R}^d$ form a closed subspace in $\mathscr{S}'$.

We conclude that (1.3.24) and (1.3.25) hold for a symmetric system when $a$ is a tempered distribution.

We use this property to define a solution when the initial data is a (not necessarily tempered) distribution. Let $a$ belong to $\mathscr{D}'(\mathbb{R}^d)^n$. Given a point $y \in \mathbb{R}^d$ and a positive number $R$, denote by $C(y;R)$ the set $y + R\mathcal{C}^-$. Choose a cut-off $\phi$ in $\mathscr{D}(\mathbb{R}^d)$, such that $\phi \equiv 1$ on $C(y,R)$. The product $\phi a$, being a compactly supported distribution, is a tempered one. Therefore, there exists a unique $u^\phi$, solution of (1.1.3) in $\mathscr{C}(\mathbb{R};\mathscr{S}')$, with initial data $\phi a$. For two choices $\phi,\psi$ of cut-off functions, $(\phi-\psi)a$ vanishes on $C(y;R)$, so that $u^\phi(t)$ and $u^\psi(t)$ coincide on $C(y;R-t)$ for $0<t<R$. This allows us to define a restriction of $u^\phi$ on the cone

$$K(y;R) := \bigcup_{0<t<R} \{t\} \times C(y;R-t).$$

As shown above, this restriction, denoted by $u_{y,R}$ does not depend on the choice of the cut-off. It actually depends only on the restriction of $a$ on $C(y;R)$. Now, if a point $(z,t)$ lies in the intersection of two such cones $K(y_1;R_1)$ and $K(y_2;R_2)$, it belongs to a third one $K(y_3;R_3)$, which is included in their intersection. The restrictions of $u_{y_1,R_1}$ and $u_{y_2,R_2}$ to $K(y_3;R_3)$ are equal, since they depend only on the restriction of $a$ on $C(y_3;R_3)$. We obtain in this way a unique distribution $u \in \mathscr{C}(\mathbb{R}^+;\mathscr{D}')$, whose restriction on every cone $K(y;R)$ coincides with $u_{y,R}$. It solves (1.1.3) in the distributional sense, and takes the value $a$ as $t=0$. Reversing the time arrow, we solve the backward Cauchy problem as well.

This construction is relevant, for instance, when $a$ is $L^2_{\rm loc}$ rather than square-integrable. It can be used also when $a$ is in $L^p_{\rm loc}$ for $p \neq 2$, even though the corresponding solutions are not $\mathscr{C}(\mathbb{R};L^p)$ in general, because of Brenner's theorem.

### 1.3.3 *Uniqueness for non-decaying data*

The construction made above, though defining a unique distribution, does not tell us about the uniqueness in $\mathscr{C}(0,T;X)$ for $a \in X$, when $X = \mathscr{D}'(\mathbb{R}^d)^n$ or $X = L^2_{\rm loc}(\mathbb{R}^d)^n$ for instance. This is because we got uniqueness results through the use of Fourier transform, a tool that does not apply here. We describe below two relevant techniques.

Let us begin with $X = L^2_{\rm loc}$. We assume that $u \in \mathscr{C}(0,T;X)$ solves (1.1.3) with $a=0$. We use the **localization** method. Let $K(y;R)$ be a cone as in the previous section, and $\phi \in \mathscr{D}(\mathbb{R}^d)$ be such that

$$\phi(x) = 1, \quad \forall x \in \bigcup_{0<t<R} C(y;R-t),$$

the latter set being the $x$-projection of $K(y;R)$. Multiplying (1.1.3) by $\phi$, and denoting $v := \phi u$, we obtain

$$\partial_t v + \sum_\alpha A^\alpha \partial_\alpha v = Bv + f,$$

where $v \in \mathscr{C}(0,T;L^2(\mathbb{R}^d)^n)$ and

$$f := (\partial_t \phi + A(\nabla_x \phi))u \in \mathscr{C}(0,T;L^2(\mathbb{R}^d)^n).$$

At this point, we are allowed to write the energy estimate

$$\partial_t |v|^2 + \sum_\alpha \partial_\alpha (A^\alpha v, v) = 2\text{Re }(Bv + f, v),$$

which gives for every $0 \le t_1 < t_2 < R$, after integration,

$$\int_{\omega(t_2)} |v(t_2)|^2 \mathrm{d}x \le \int_{\omega(t_1)} |v(t_1)|^2 \mathrm{d}x + \int_{t_1}^{t_2} \mathrm{d}t \int_{\omega(t)} 2\text{Re }((Bv,v) + (f,v))\mathrm{d}x, \tag{1.3.27}$$

where $\omega(t) := C(y; R-t)$. However, the equalities $v = u$, $f = 0$ hold in $K(y;R)$. Therefore (1.3.27) reduces to

$$\int_{\omega(t_2)} |u(t_2)|^2 \mathrm{d}x \le \int_{\omega(t_1)} |u(t_1)|^2 \mathrm{d}x + 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\omega(t)} \text{Re }(Bu,u)\,\mathrm{d}x.$$

This, with the Gronwall inequality, gives

$$\int_{\omega(t)} |u(x,t)|^2 \mathrm{d}x \le e^{2t\|B\|} \int_{\omega(0)} |u(x,0)|^2 \mathrm{d}x = 0.$$

Since $y$ and $R$ are arbitrary, we obtain $u \equiv 0$ almost everywhere, which is the uniqueness property.

We now turn to the case $X = \mathscr{D}'(\mathbb{R}^d)^n$, where the former argument does not work. Our main ingredient is the *Holmgren principle*, a tool that we shall develop more systematically in subsequent chapters. We assume that $u \in \mathscr{C}(0,T;X)$ solves (1.1.3) in the distributional sense. This means that, for every test function $\phi \in \mathscr{D}(\mathbb{R}^d \times (0,T))^n$, it holds that

$$\langle u, L^* \phi \rangle = 0, \quad L^* := -\partial_t - \sum_\alpha (A^\alpha)^T \partial_\alpha - B^T.$$

This may be rewritten as

$$\int_0^T \langle u(t), L^*\phi(t)\rangle\,\mathrm{d}t = 0. \tag{1.3.28}$$

Let $\psi$ be a slightly more general test function: $\psi \in \mathscr{D}(\mathbb{R}^d \times (-\infty, T))^n$. Choosing $\theta \in \mathscr{C}^\infty(\mathbb{R})$ with $\theta(\tau) = 0$ for $\tau < 1$ and $\theta(\tau) = 1$ for $\tau > 2$, we define

$$\phi_\epsilon(x,t) = \theta(t/\epsilon)\psi(x,t).$$

We may apply (1.3.28) to $\phi_\epsilon$, which gives

$$\int_0^T \theta(t/\epsilon)\langle u(t), L^*\psi(t)\rangle \,\mathrm{d}t = \frac{1}{\epsilon}\int_0^T \theta'(t/\epsilon)\langle u(t), \psi(t)\rangle \,\mathrm{d}t.$$

Using the continuity in time, we may pass to the limit as $\epsilon \to 0^+$, and obtain

$$\int_0^T \langle u(t), L^*\psi(t)\rangle \,\mathrm{d}t = \langle u(0), \psi(0)\rangle.$$

Therefore, assuming $u(0) = 0$, we see that (1.3.28) is valid for $\psi$ as well, that is to test functions in $\mathscr{D}(\mathbb{R}^d \times (-\infty, T))^n$.

We now choose an arbitrary test function $f \in \mathscr{D}(\mathbb{R}^d \times (0,T))^n$. Obviously, $L^*$ is a hyperbolic operator and we can solve the backward Cauchy problem

$$L^*\chi = f, \quad \chi(T) = 0.$$

Extending $f$ by zero for $t \le 0$, we obtain a unique solution $\chi \in \mathscr{C}^\infty(-\infty, T; \mathscr{S})$. Applying (1.3.26) to this backward problem, we see that $\chi(t)$ has compact support for each time, with $\operatorname{Supp}\chi(t)$ included in a ball of the form $B_{\rho(T-t)}$, for a suitable constant $\rho$. Also, $\chi$ vanishes identically for $t$ close enough to $T$ (because $f$ does). Truncating, we apply (1.3.28) to $\psi(x,t) = \theta(t+1)\chi(x,t)$. This gives $\langle u, f\rangle = 0$ for all test functions, that is $u = 0$. Therefore the Cauchy problem for a Friedrichs-symmetric operator has the uniqueness property in the class $\mathscr{C}(0,T;\mathscr{D}')$.

## 1.4 Directions of hyperbolicity

The situation for general (weakly) hyperbolic operators is not as neat as that for Friedrichs-symmetrizable ones. Non-symmetrizable operators do exist, as soon as $d = 2$ and $n = 3$, as shown by Lax [110]. The class of constantly hyperbolic operators provides a valuable and flexible alternative to Friedrichs-symmetrizable ones. Their analysis will lead us to several new and useful notions.

In this section, we shall not address the problem of propagation of the support (with finite velocity), which we solved in the symmetric case. This propagation holds true for constantly hyperbolic systems, but a rigorous proof needs a theory of the Cauchy problem for systems with variable coefficients. Such a theory will be done in Chapter 2, where we shall prove an accurate result.

### 1.4.1 *Properties of the eigenvalues*

The results in this section are essentially those of Lax [110], and the arguments follow Weinberger [217], though we give a more detailed proof of the claim below.

We begin by considering a subspace $E$ in $\mathbf{M_n}(\mathbb{R})$, with the property that every matrix in $E$ has a real spectrum. Without loss of generality, we may assume that $I_n$ belongs to $E$. If $M \in E$, we denote by $\lambda_1(M) \le \cdots \le \lambda_n(M)$ the spectrum of $M$, counting with multiplicities. The functions $\lambda_j$ are positively homogeneous of order one. They are continuous, but could be non-differentiable at crossing points. In the constantly hyperbolic case, however, they are analytic away from the origin.

**Lemma 1.1** *Let $A$ and $B$ be matrices in $E$, with $\lambda_1(B) > 0$. Then the eigenvalues of $B^{-1}(\lambda I_n - A)$ are real.*

**Proof** From the assumption, we know that $B$ is non-singular. Define a polynomial

$$P(X, Y) := \det(XI_n - A - YB),$$

which has degree $n$ with respect to $X$ as well as to $Y$. Define continuous functions $\phi_j(\mu) = \lambda_j(A + \mu B)$. From homogeneity and continuity, we have

$$\phi_j(\mu) \sim \begin{cases} \mu\lambda_j(B), & \text{as } \mu \to +\infty, \\ \mu\lambda_{n+1-j}(B) & \text{as } \mu \to -\infty. \end{cases}$$

Hence $\phi_j(\mu)$ tends to $\pm\infty$ with $\mu$. By the Intermediate Value Theorem, it must take any prescribed real value $\lambda$ at least once.

Thus, let $\lambda^*$ be given and $\mu_j \in \mathbb{R}$ be a root of $\phi_j(\mu_j) = \lambda^*$ for each $j$. Given one of these roots, $\mu^*$, let $J$ be the number of indices such that $\mu_j = \mu^*$. Then $\lambda^*$ is a root of $P(\cdot, \mu^*)$, of order $J$ at least.

**Claim 1.1** *The multiplicity of $\mu^*$ as a root of $P(\lambda^*, \cdot)$ is larger than or equal to $J$.*

This claim readily implies the lemma. Its proof is fairly simple when the $\phi_j$s are differentiable, for instance in the constantly hyperbolic case. But in the general case, one must use once more the assumption. To simplify the notations, we assume without loss of generality that $\lambda^* = \mu^* = 0$, by translating $A$ to $A + \mu^* B - \lambda^* I_n$. Let $N$ ($N \geq J$) be the multiplicity of the null root of $P(\cdot, 0)$. The Newton's polygon of the polynomial $P$ admits the vertices $(N, 0)$ and $(0, K)$.

Let $\delta$ be the edge of the Newton's polygon with vertex $(N, 0)$. We denote its other vertex by $(j, k)$. Retaining only those monomials of $P$ whose degrees $(a, b)$ belong to $\delta$, we obtain a polynomial $X^j Q$ with the following homogeneity:

$$Q(a^k X, a^{N-j} Y) = a^{k(N-j)} Q(X, Y).$$

It is a basic fact in algebraic geometry (see [35], Section 2.8) that, in the vicinity of the origin, the algebraic curve $P(x, y) = 0$ is described by simpler curves corresponding to the edges of the Newton polygon, up to analytic diffeomorphisms. In the present case, these diffeomorphisms have real coefficients (i.e. they preserve real vectors) since $P$ has real coefficients. The 'simple' curve $\gamma$ associated to $\delta$ is just that with equation $Q(x, y) = 0$. Hence, points $(x, y)$ in $\gamma$ with a real co-ordinate $y$ must be real (because this is so in the curve $P = 0$.)

Let $\omega$ be a root of unity, of order $2(N - j)$, that is $\omega^{N-j} = -1$. Because of the homogeneity, the map $(x, y) \mapsto (\omega^k x, -y)$ preserves $\gamma$. If $y$ is real, the map thus moves a real point into another one. Hence, $\omega^k$ is real, thus $\omega^{2k} = 1$. This implies that $k$ is a multiple of $N - j$. In particular, $k \geq N - j$.

Since $(j, k)$ is a vertex of the Newton polygon, lying between the vertices $(N, 0)$ and $(0, K)$, we have

$$\frac{j}{N} + \frac{k}{K} \leq 1.$$

Together with $k \geq N - j$, this implies $K \geq N$ and the claim. □

Suppose that in the proof of Lemma 1.1, one of the functions, say $\phi_l$, is not strictly monotone. For a suitable real number $\lambda$, the equation $\phi_l(\mu) = \lambda$ will have at least three roots, and $P(\lambda, \cdot) = 0$ will have $n + 2$ roots at least, which is absurd. Therefore, the assumption $\lambda_1(B) > 0$ implies that $\mu \mapsto \lambda_j(A + \mu B)$ is monotone increasing. For a general $B$ in $E$ we may apply that to $B' := B - cI_n$ with $c < \lambda_1(B)$. Letting $c \to \lambda_1(B)$, we obtain that

$$\mu \to \lambda_j(A + \mu B) - \mu\lambda_1(B)$$

is non-decreasing. In particular,

$$\lambda_j(A + B) \geq \lambda_j(A) + \lambda_1(B), \quad \forall A, B \in E. \tag{1.4.29}$$

Reversing $(A, B)$ into $(-A, -B)$, we also have

$$\lambda_j(A + B) \leq \lambda_j(A) + \lambda_n(B), \quad \forall A, B \in E. \tag{1.4.30}$$

In particular, with $j = 1$ in (1.4.29) and $j = n$ in (1.4.30), we obtain:

**Proposition 1.5** *Let $E$ be a vector space of real $n \times n$ matrices, whose every element has a real spectrum.*

*The smallest eigenvalue is a concave function, while the largest is a convex one: For every $A$ and $B$ in $E$,*

$$\lambda_1(A + B) \geq \lambda_1(A) + \lambda_1(B),$$
$$\lambda_n(A + B) \leq \lambda_n(A) + \lambda_n(B).$$

*This applies to the space $E := \{A(\xi)\, ;\, \xi \in \mathbb{R}^d\}$ when the operator $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is hyperbolic.*

**Remark** When $E = \mathbf{Sym_n}(\mathbb{R})$, a space that obviously satisfies the assumption, (1.4.29) and (1.4.30) belong to the set of Weyl's inequalities. Given the spectra of $A$ and $B$, but not $A$ and $B$ themselves, Horn's conjecture, proved recently by Klyachko [98] and Knutson and Tao [99], characterizes the set of possible spectra of $A + B$ as a convex polytope, defined through rather involved linear inequalities. It would be interesting to know† which of these inequalities remain

†This question has been solved recently, thanks to the efforts of J. Helton, V. Vinnikov and L. Gurvits. It turns out that every linear inequality that is valid for real symmetric matrices is valid for matrices in $E$. These inequalities actually apply to the roots of an arbitrary hyperbolic homogeneous polynomial.

true for a general subspace $E$ treated in this section. The simplest ones in Horn's conjecture are Weyl's inequalities

$$\lambda_k(A+B) \leq \lambda_i(A) + \lambda_j(B), \qquad (k+n \leq i+j),$$

and

$$\lambda_k(A+B) \geq \lambda_i(A) + \lambda_j(B), \qquad (k+1 \geq i+j).$$

Next comes the Theorem of Lidskii, which tells us that, as a vector in $\mathbb{R}^n$, the spectrum of $A+B$ belongs to the convex hull of $(P^\sigma)_{\sigma\in\mathbf{S}_n}$ where $P_\sigma$ has coordinates $\lambda_i(A) + \lambda_{\sigma(i)}(B)$ and $\sigma$ runs over all permutations. See Exercises 11 of Chapter 3 and 19 of Chapter 5 in [187].

Lemma 1.1 can be improved in the following way. Given $\lambda^* \in \mathbb{R}$, let $\sigma_1, \dots, \sigma_s$ be the distinct eigenvalues of $M = B^{-1}(\lambda I_n - A)$. Let $S_\ell$ be the set of indices $j$ such that $\mu_j(\lambda^*) = \sigma_\ell$ and $J_\ell$ its cardinality. Since each function $\phi_j$ is strictly monotone, we have $\lambda^* - \phi_j(\sigma_\ell) \neq 0$ for every $j$ not in $S_\ell$. Therefore, $J_\ell$ is precisely the multiplicity of the root $\lambda^*$ of $P(\cdot, \sigma_\ell)$. From the claim, we know that $J_\ell$ is less than or equal to the multiplicity $m_\ell$ of $\sigma_\ell$ as a root of $P(\lambda^*, \cdot)$. Hence

$$n = J_1 + \cdots + J_s \leq m_1 + \cdots + m_s = n,$$

and we conclude that $m_\ell = J_\ell$ for each $\ell$.

**Lemma 1.2** *With the assumptions of Lemma 1.1, let a real pair satisfy* $P(\lambda, \mu) = 0$*, where* $P(X, Y) := \det(XI_n - A - YB)$*. Then the multiplicities of* $\lambda$ *as a root of* $P(\cdot, \mu)$*, and of* $\mu$ *as a root of* $P(\lambda, \cdot)$*, coincide.*

Finally, we remark that $A \mapsto \max\{-\lambda_1(A), \lambda_n(A)\}$ is a semi-norm over such a space $E$ as above.

### 1.4.2 *The characteristic and forward cones*

From now on, $E$ is the set of matrices $\tau I_n + A(\xi)$ for $(\tau, \xi) \in \mathbb{R}^{1+d}$, where $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is a hyperbolic operator.

**Definition 1.5** *The* characteristic cone *of the hyperbolic operator* $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ *is the set*

$$\mathrm{char}L := \{(\xi, \lambda) \in \mathbb{R}^d \times \mathbb{R}\,;\, \det(A(\xi) + \lambda I_n) = 0\}.$$

*Its elements are the* characteristic frequencies*. The connected component of* $(0, 1)$ *in* $(\mathbb{R}^d \times \mathbb{R}) \setminus \mathrm{char}L$ *is denoted by* $\Gamma$*; it is called the* forward cone.

Obviously, $\Gamma$ is a kind of epigraph of $\lambda_n$:

$$\Gamma = \{(\xi, \lambda)\,;\, \lambda > \lambda_n(-\xi)\}.$$

According to Proposition 1.5, it is a convex cone in $\mathbb{R}^{d+1}$, a result originally due to Gårding [65, 66]. The terminology *forward cone* will be explained in the next section.

When $L$ is constantly hyperbolic, the eigenvalues $\lambda_j$ are analytic away from the origin. The function $\lambda_n$ has therefore a non-negative Hessian $\mathbf{D}^2\lambda_n$. Because of homogeneity, this Hessian is indefinite,

$$\mathbf{D}^2\lambda_n(\xi)\xi = 0.$$

Therefore $\Gamma$ is not strictly convex in the usual sense, and we shall say that $\lambda_n$ is *transversally strictly convex* if the equality

$$\theta\lambda_n(\xi) + (1-\theta)\lambda_n(\xi') = \lambda_n(\theta\xi + (1-\theta)\xi'), \quad \theta \in (0,1)$$

implies $\xi' \in \mathbb{R}^+\xi$. We now prove that such a strict convexity holds for most systems.

**Proposition 1.6** *Let the operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

*be hyperbolic. Then the forward cone* $\Gamma$ *is convex.*

*If* $L$ *is constantly hyperbolic, then either the function* $\lambda_n$ *is transversally strictly convex (and therefore* $\lambda_1$ *is transversally strictly concave), or the system is a vector-valued transport equation*

$$\partial_t u + (\vec{V}\cdot)\nabla_x u = 0.$$

.

**Proof** If $\lambda_n$ is not transversally strictly convex, there exists a segment $[\xi_1, \xi_2]$ on which $\lambda_n$ is affine, and $\xi_1, \xi_2$ are not parallel. By homogeneity, $\lambda_n$ is affine on the triangle with vertices $0, \xi_1, \xi_2$. Since $\lambda_n(0) = 0$, 'affine' actually means 'linear'. Let $P$ be the plane spanned by $\xi_1, \xi_2$. Since $\lambda_n$ is analytical away from the origin, and since $P \setminus \{0\}$ is a connected set, the restriction of $\lambda_n$ to $P$ is linear. It follows that $\lambda_n(\xi_1) = -\lambda_n(-\xi_1)$. In other words, $\lambda_n(\xi_1) = \lambda_1(\xi_1)$. This means that $A(\xi_1)$ has only one eigenvalue. Finally, the system being constantly hyperbolic, there must be only one eigenvalue for every $\xi$. Since $A(\xi)$ is diagonalizable, this gives $A(\xi) = \lambda_n(\xi)I_n$. Therefore, $\lambda_n(\xi) = \mathrm{Tr}\, A(\xi)/n$, which shows that $\lambda_n$ is linear on the whole $\mathbb{R}^d$, thus there exists a vector $\vec{V}$ such that $\lambda_n(\xi) = \vec{V}\cdot\xi$. This ends the proof. □

### 1.4.3 *Change of variables*

The role of the cone $\Gamma$ becomes clear when we consider changes of the space–time reference frame. Let us perform a linear change of independent variables

$$(x,t) \mapsto (y,s), \quad y = Rx + tV, \quad s = \lambda_0 t + \xi_0 \cdot x,$$

with $R \in M_d(\mathbb{R})$ and $V, \xi_0 \in \mathbb{R}^d$, chosen so that the whole matrix

$$\mathcal{R} := \begin{pmatrix} R & V \\ \xi_0^T & \lambda_0 \end{pmatrix}$$

is invertible. The system (1.1.3) is changed into

$$\frac{\partial u}{\partial s} + \sum_\alpha \widetilde{A}^\alpha \frac{\partial u}{\partial y_\alpha} = \widetilde{B} u,$$

where

$$\widetilde{A}^\alpha := (\lambda_0 I_n + A(\xi_0))^{-1} \left( \sum_\beta R_{\alpha\beta} A^\beta + V_\alpha I_n \right), \tag{1.4.31}$$

provided that $(\xi_0, \lambda_0)$ is not characteristic. We consider the variable $s$ as a new time variable and look at the Cauchy problem. Let us point out that it is not equivalent to the former Cauchy problem, since the data is now given on the hyperplane $\{s = 0\}$, instead of $\{t = 0\}$. Its strong well-posedness is equivalent to the hyperbolicity of the operator

$$\frac{\partial}{\partial s} + \sum_\alpha \widetilde{A}^\alpha \frac{\partial}{\partial y_\alpha}.$$

A change of variables that preserves $t$ (that is with $\xi_0 = 0$, $\lambda_0 = 1$) is harmless, giving $\widetilde{A}(\eta) = A(\xi) + (\xi \cdot R^{-1} V) I_n$ with $\xi = R^T \eta$, so that hyperbolicity is preserved. Therefore, hyperbolicity is really a property of the pair $(\xi_0, \lambda_0)$, which determines the direction of the hyperplane $\{s = 0\}$ where the Cauchy data is given. This leads us to the following.

**Definition 1.6** *We say that the operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

*is* hyperbolic in the direction $(\xi_0, \lambda_0)$, *if* $(\xi_0, \lambda_0)$ *is not characteristic, and if, moreover, the operator*

$$\widetilde{L} := \frac{\partial}{\partial s} + \sum_\alpha \widetilde{A}^\alpha \frac{\partial}{\partial y_\alpha} \tag{1.4.32}$$

*is hyperbolic, with* $\widetilde{A}^\alpha$ *being defined in (1.4.31).*

**Remarks**

- In particular, $\partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is hyperbolic in the direction $(0, 1)$ if and only if it is hyperbolic in the sense that we considered so far.
- The hyperbolicity in directions $(\xi_0, \lambda_0)$ and $(-\xi_0, -\lambda_0)$ are equivalent. Therefore, we may always restrict ourselves to $\lambda_0 \geq 0$.
- When $\partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is symmetric, and when $\lambda_0 I_n + A(\xi_0)$ is positive-definite, the new operator $\partial_s + \sum_\alpha \widetilde{A}^\alpha \partial_\alpha$ is Friedrichs symmetrizable, with $\lambda_0 I_n + A(\xi_0)$ as a symmetrizer. This statement is to be compared with

Theorem 1.5 below, with the remark that the positive-definiteness means that $\lambda_0 I_n + A(\xi_0)$ belongs to $\Gamma$ in this case.

- If $(\xi_0, \lambda_0) \in \text{char}L$, the matrices $\widetilde{A}^\alpha$ are not well-defined; the variable $s$ cannot be taken as a time variable. In this case, we say that the hyperplane $\{s = 0\}$ is *characteristic*. We shall come back to this important notion later.

Hyperbolicity in the direction $(\xi_0, \lambda_0)$ means that the matrices

$$\widetilde{A}(\eta) := (\lambda_0 I_n + A(\xi_0))^{-1} \left(A(R^T\eta) + (V \cdot \eta)I_n\right)$$

have a real spectrum for every $\eta \in \mathbb{R}^d$, and are uniformly diagonalizable. Lemma 1.1 tells us that this spectrum is real for every $\eta \in \mathbb{R}^d$, as soon as $\lambda_1(\lambda_0 I_n + A(\xi_0)) > 0$, which means $\lambda_0 + \lambda_1(A(\xi_0)) > 0$. On the other hand, one has, with the notations of Lemma 1.1,

$$\ker(B^{-1}(\lambda I_n - A) - \mu I_n) = \ker(\lambda I_n - A - \mu B).$$

Since now every element of $E$ is diagonalizable, the dimension of the right-hand side equals the multiplicity of $\lambda$ as a root of $P(\cdot, \mu)$. From Lemma 1.2, we deduce that the dimension of the left-hand side equals the multiplicity of $\mu$ as a root of $P(\lambda, \cdot)$. Hence the algebraic and geometric multiplicities coincide: $B^{-1}(\lambda - A)$ is diagonalizable. Applying this result to the above context, we conclude that $\widetilde{A}(\eta)$ is diagonalizable with a real spectrum, for every $\eta \in \mathbb{R}^d$. We leave the reader to verify that the diagonalization can be performed uniformly, using the assumption that it is true in $E$.

In two instances, the verification of this fact is rather easy. For, if $L$ is symmetric and $(\lambda_0, \xi_0) \in \Gamma$, then $\lambda_0 I_n + A(\xi_0)$ is positive-definite and plays the role of a symmetrizer for $\tilde{L}$. On the other hand, assume that $L$ is strictly hyperbolic (or more generally constantly hyperbolic). Looking back at the proof of Lemma 1.1, the functions $\phi_j$ and $\phi_k$ cannot coincide somewhere if $j \neq k$. Hence $B^{-1}(\lambda I_n - A)$ has distinct eigenvalues. It follows that $\tilde{L}$ is strictly (or constantly) hyperbolic too.

Therefore, we have the following result.

**Theorem 1.5** *A hyperbolic operator $L$ is hyperbolic in every direction of its forward cone. If $L$ is either Friedrichs symmetrizable, or strictly, or constantly hyperbolic, then $L$ has the same property in every direction of its forward cone.*

**Comments** It is known that when $E$ is a subspace of $\mathbf{M_n}(\mathbb{R})$, consisting only on diagonalizable matrices with real eigenvalues, these eigenvalues may be labelled, at least locally, with the property that one-sided directional derivatives

$$\lim_{h \to 0^+} \frac{\lambda(T + hT_1) - \lambda(T)}{h} =: \delta\lambda_T(T_1)$$

exist. However, $\delta\lambda_T(T_1)$ may be neither linear with respect to $T_1$, nor continuous in $T$. Although it is positively homogeneous in $T_1$, it may not satisfy

$$\delta\lambda_T(-T_1) = -\delta\lambda_T(T_1). \tag{1.4.33}$$

We illustrate these facts with a two-dimensional space, spanned by the matrices

$$A^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix $\xi_1 A^1 + \xi_2 A^2$ has eigenvalues $\pm|\xi|$. We choose $\lambda_1 = -|\xi|$, $\lambda_2 = |\xi|$. These functions are not differentiable at the origin but have the one-sided directional derivatives mentioned above. There is no way to relabel the eigenvalues in order to satisfy (1.4.33). A more involved example is the set of symmetric $n \times n$ matrices.

A rather complete analysis of these facts may be found in Chapter 2 of Kato's book [95]. It also contains (Theorem 5.4 of [95]) the following amazing fact, which shows that the two-dimensional example above is optimal. When restricting to a curve $s \mapsto T(s)$ in $E$, where the parametrization is differentiable (respectively, analytic), one may label the eigenvalues in such a way that they are differentiable (respectively, analytic) with respect to $s$. In other words, one may satisfy (1.4.33) when there is only one scalar parameter, though it will be at the price of a loss of ordering.

### 1.4.4 *Homogeneous hyperbolic polynomials*

The theory of scalar equations of higher order involves the notion of *hyperbolic polynomials*. Let $p$ be a homogeneous polynomial of degree $n$ in $d+1$ variables $\xi_0, \dots, \xi_d$. We consider the equation

$$p\left(\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_d}\right) u = f.$$

According to Gårding [65], we say that $p$ is hyperbolic in the direction of some real vector $a \in \mathbb{R}^{d+1}$ if for every vector $\xi \in \mathbb{R}^{d+1}$, the equation

$$p(\tau a + \xi) = 0$$

has $n$ real roots, counting multiplicities. This implies $p(a) \neq 0$ and we may normalize $p$ by $p(a) = 1$. Notice that the traditional case where $x_0 = t$ and $p$ is hyperbolic in the direction of time corresponds to $a = (1, 0, \dots, 0)$. A typical example is $p(\xi) = \xi_0^2 - \xi_1^2 - \cdots - \xi_d^2$, which is associated to the wave operator $\partial_t^2 - \Delta_x^2$, and is hyperbolic in the 'direction of time' $a = (1, 0 \dots, 0)$.

The definition of hyperbolicity given above is equivalent to the $\mathcal{C}^\infty$ well-posedness of the Cauchy problem for the equation

$$p(\partial_0, \dots, \partial_d) u = f.$$

However, it does not imply $L^2$- or $H^s$-well-posedness (in a sense adapted to the order of the operator); it is merely the analogue of the weak hyperbolicty described in Section 1.1. We refer to [65] for the case where $p$ is not homogeneous. Gårding's definition of hyperbolicity is the more general one, and extends, for instance, that of Petrowsky [158].

We shall not discuss here the Cauchy problem for general hyperbolic operators. This has given rise to an enormous literature. However, we do not resist to mention the remarkable convexity results obtained by Gårding in [66]. The first property is that the polynomial $q$, homogeneous of degree $n-1$, defined by

$$q(\xi) := \sum_{\alpha=0}^{d} a_\alpha \frac{\partial p}{\partial \xi_\alpha}$$

is hyperbolic in the direction of $a$ too. This is the interlacing property of real zeroes of a univariate polynomial and its derivative. Let us give an immediate application. It is clear that a linear form is hyperbolic in every non-characteristic direction, and also that the product of polynomials that are hyperbolic in some direction $a$ (the same for every one), is hyperbolic also in this direction. For instance, $\sigma_{d+1}(\xi) := \prod_\alpha \xi_\alpha$ is hyperbolic in the direction of $(1, \dots, 1)$. Applying repeatedly the derivation in direction $a$, we deduce that every elementary symmetric polynomial $\sigma_k(\xi)$ is hyperbolic in the direction $(1, \dots, 1)$. This is trivial if $k = 1$ (pure transport), and this is well known if $k = 2$, because $\sigma_2$ is a quadratic form of index $(1, d)$, positive on $(1, \dots, 1)$.

The forward cone $C_p(a)$ is the connected component of $a$ in the set defined by $p(\xi) > 0$. As in the case of first-order systems, $C(a)$ is convex, and $p$ is hyperbolic in the direction of $b$ for every $b$ in $C_p(a)$. If $q$ is the $a$-derivative as above, then $C_p(a) \subset C_q(a)$, with obvious notation.

The nicest result is perhaps the following. Let $P$ be the polarized form of $p$, meaning that

$$(\xi^1, \dots, \xi^n) \mapsto P(\xi^1, \dots, \xi^n)$$

is a symmetric multilinear form, such that $P(\xi, \dots, \xi) = p(\xi)$ for every $\xi \in \mathbb{R}^{1+d}$ (this is the generalization of the well-known polarization of a quadratic form). Then we have

$$\left(\xi^1 \in C_p(a), \dots, \xi^n \in C_p(a)\right) \Longrightarrow \left(p(\xi^1) \cdots p(\xi^n) \le P(\xi^1, \dots, \xi^n)^n\right). \quad (1.4.34)$$

We point out that when $n = 2$, that is when $p$ is a quadratic form of index $(1, d)$, this looks like the Cauchy–Schwarz inequality, except that (1.4.34) is in the opposite sense. An equivalent statement is that

$$\xi \mapsto p(\xi)^{1/n}$$

is a concave function over $C_p(a)$.

Gårding's results have had many consequences in various fields, including differential geometry, elliptic (!) PDEs (see, for instance, the article by Caffarelli *et al.* [29]) and interior point methods in optimization. A rather surprising byproduct is the concavity of the function[6]

$$H \mapsto (\det H)^{1/n}, \qquad H \in \mathbf{HPD}_n.$$

This property is reminiscent of the Alexandrov–Fenchel inequality

$$\mathrm{vol}(K_1)\mathrm{vol}(K_2) \leq V(K_1, K_2)^2$$

for convex bodies, where $V$ denotes the *mixed volume*. The van der Waerden inequality for the permanent of a doubly stochastic matrix can be rewritten in terms of an inequality for hyperbolic polynomials, applied to $\sigma_n$ in $n$ indeterminates.

## 1.5 Miscellaneous

### 1.5.1 *Hyperbolicity of subsystems*

Let $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be a hyperbolic $n \times n$ operator. Given a linear subspace $G$ of $\mathbb{R}^n$ of dimension $m$, with a projector $\pi$ onto $G$, one may form a subsystem in $m$ unknowns $v(x,t) \in G$ and $m$ equations, governed by the operator $L' = \partial_t + \sum_\alpha \pi A^\alpha \partial_\alpha$. There is no reason, in general, why $L'$ would be hyperbolic. The following result shows that a clever choice of $\pi$ ensures this hyperbolicity.

**Theorem 1.6** *Let $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be a hyperbolic $n \times n$ operator and $\xi_0$ belong to $S^{d-1}$. Given an eigenvalue $\lambda_0$ of the spectrum of $A(\xi_0)$, denote by $\pi$ the eigenprojection onto the eigenspace $F(\lambda_0) := \ker(A(\xi_0) - \lambda_0 I_n)$.*

*Then the operator*

$$L' := \partial_t + \sum_{\alpha=1}^{d} \pi A^\alpha \partial_\alpha,$$

acting on functions valued in $F(\lambda_0)$ *(thus it is an $m \times m$ operator, $m$ being the multiplicity of $\lambda_0$), is hyperbolic.*

This result is of low interest when $L$ is symmetric hyperbolic (or more generally smmetrizable), for then $\pi$ is an orthogonal projection, so that $\pi A(\xi) : F(\lambda_0) \to F(\lambda_0)$ is symmetric, thus $L'$ is symmetric hyperbolic too.

**Proof** Using a linear change of unknowns, which amounts to conjugating the matrices $A^\alpha$, we may assume that $A(\xi_0)$ is diagonal:

$$A(\xi_0) = \begin{pmatrix} \lambda_0 I_m & 0 \\ 0 & D_0 \end{pmatrix},$$

[6] This result is strictly better than the well-known concavity of $H \mapsto \log\det H$ for positive-definite Hermitian matrices. However, this latter statement has the advantage of having a form independent of $n$.

where $D_0 - \lambda_0 I_{n-m}$, of size $n-m$, is invertible. We decompose vectors and matrices accordingly:

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A^\alpha = \begin{pmatrix} C^\alpha & F^\alpha \\ E^\alpha & D^\alpha \end{pmatrix}.$$

The theorem states that the $m \times m$ operator

$$L' := \partial_t + \sum_{\alpha=1}^{d} C^\alpha \partial_\alpha$$

is hyperbolic.

Since $\mathbb{C}^n$ is the direct sum $(\mathbb{R}^m \times \{0_{n-m}\}) \oplus (\{0_m\} \times \mathbb{R}^{n-m})$ of invariant subspaces of $A(\xi_0)$, corresponding to disjoint parts of the spectrum, standard perturbation theory (see Kato [95]) tells us that there exists a neighbourhood $\mathcal{V}$ of $\xi_0$ and an analytical map $\xi \mapsto K(\xi)$ from $\mathcal{V}$ to $M_{(n-m)\times m}(\mathbb{R})$, such that

*i)* $K(\xi_0) = 0$,

*ii)* the subspace

$$N(\xi) := \left\{ \begin{pmatrix} x \\ K(\xi)x \end{pmatrix} ; \, x \in \mathbb{R}^m \right\}$$

is invariant under $A(\xi)$.

Hence, $N(\xi)$ is invariant under the flow of $\dot{X} = A(\xi)X$. On this subspace, the flow is defined by $\dot{x} = Q(\xi)x$, $y = K(\xi)x$, where

$$Q(\xi) := C(\xi) + F(\xi)K(\xi).$$

Let us define

$$M := \sup_\xi \| \exp iA(\xi) \|,$$

which is finite by assumption. For every $\xi$ in $\mathcal{V}$, $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^m$, it holds that

$$\| \exp(itQ(\xi))x_0 \| \leq c_0 M (\|x_0\| + \|K(\xi)x_0\|),$$

where $c_0$ is accounted for the equivalence of the standard norm with $(x, K(\xi)x) \mapsto \|x\|$, on $N(\xi)$. In other words,

$$\| \exp(itQ(\xi)) \| \leq c_0 M (1 + \|K(\xi)\|). \tag{1.5.35}$$

Let $\eta$ be given in $\mathbb{R}^d$. One applies (1.5.35) to the vector $\xi = \xi_0 + s\eta$, for $s$ small enough (in such a way that $\xi \in \mathcal{V}$) and $t = 1/s$. Since

$$Q(\xi) = \lambda_0 I_m + sC(\eta) + sF(\eta)K(\xi),$$

it holds that

$$\exp itQ(\xi) = \mathrm{e}^{it\lambda_0} \exp i(C(\eta) + F(\eta)K(\xi)).$$

Using (1.5.35) and passing to the limit as $s \to 0$, we obtain

$$\| \exp iC(\eta) \| \le c_1 M,$$

which proves the claim. □

We improve now Theorem 1.6 for constantly hyperbolic systems. Theorem 1.7 below is attributed to Lax. It turns out to be useful in geometrical optics in the presence of non-simple eigenvalues. It will also be valuable in the study of characteristic initial boundary value problems, see Section 6.1.3.

To begin with, we consider a constantly hyperbolic operator $L$ and select an eigenvalue $\lambda(\xi)$, whose multiplicity, for $\xi \neq 0$, is denoted by $m$. Denote by $\pi_\xi$ the eigenprojector onto $\ker(A(\xi) - \lambda(\xi) I_n)$. Obviously, $\lambda$ and $\pi$ are analytic functions on $\mathbb{R}^d \setminus \{0\}$.

**Theorem 1.7** *Assume that $L$ is constantly hyperbolic and adopt the above notations. Then, for every $\xi \neq 0$ and every $\eta \in \mathbb{R}^d$, it holds that*

$$\pi_\xi A(\eta) \pi_\xi = (\mathrm{d}\lambda(\xi) \cdot \eta) \pi_\xi.$$

**Proof** Differentiating the identity $(A(\xi) - \lambda(\xi))\pi_\xi = 0$, we obtain

$$(A(\xi) - \lambda(\xi))(\mathrm{d}\pi(\xi) \cdot \eta) + (A(\eta) - \mathrm{d}\lambda(\xi) \cdot \eta)\pi_\xi = 0.$$

We eliminate the factor $\mathrm{d}\pi(\xi) \cdot \eta$ by multiplying this equality by $\pi_\xi$ on the left. □

In matrix terms, we may choose co-ordinates in $\mathbb{R}^n$ such that, for some vector $\xi \neq 0$,

$$A(\xi) = \begin{pmatrix} \lambda(\xi) I_m & 0 \\ 0 & A' \end{pmatrix}, \quad \det(A' - \lambda(\xi) I_{n-m}) \neq 0.$$

The theorem above tells us that if $\lambda$ has a constant multiplicity, one has

$$A(\eta) = \begin{pmatrix} (\eta \cdot X) I_m & B(\eta) \\ C(\eta) & D(\eta) \end{pmatrix}, \quad \forall \eta \in \mathbb{R}^d,$$

for some vector $X \in \mathbb{R}^d$.

**Corollary 1.1** *Let $L$ be constantly hyperbolic, with an eigenvalue $\lambda$ of multiplicity $m > n/2$. Then $\xi \mapsto \lambda(\xi)$ is linear.*

**Proof** From the assumption, there exists a non-zero vector $x$ in the intersection of $\ker(A(\xi) - \lambda(\xi))$ and $\ker(A(\eta) - \lambda(\eta))$. On the one hand, $\pi_\xi x = x$. On the other hand, $A(\eta)x = \lambda(\eta)x$. Applying Theorem 1.7 gives $\lambda(\eta) = \mathrm{d}\lambda(\xi) \cdot \eta$. □

**Remarks**

- The example given in Section 1.2.3 shows that assuming only the diagonalizability on $\mathbb{R}^n$ of all matrices $A(\xi)$ does not ensure the hyperbolicity of the suboperator $L'$, since one of the matrices $C(\eta)$ is a Jordan block $J(0;2)$ (take $d=2$, $n=3$, $\xi_0=\vec{e}^2$ and $\lambda_0=0$).

- The assumption of constant hyperbolicity in Theorem 1.7 may be relaxed by assuming only hyperbolicity with an eigenvalue $\sigma \mapsto \lambda(\sigma)$ of constant multiplicity in the neighbourhood of $\xi$.
- The conclusion in Theorem 1.7 may not be true when we drop the assumption of constant multiplicity. For instance, let us consider a symmetric hyperbolic operator $L$. We may assume that $\xi=\vec{e}^d$ and that $N(\xi)$ equals $\mathbb{R}^m \times \{0\}$. In other words, $A^d$ is block-diagonal with the last block equal to $\lambda I_{n-m}$. Then $\pi_\xi A(\eta)\pi_\xi$ is the first diagonal block of $A(\eta)$. It may be any linear map into the space of real symmetric $m \times m$ matrices. A refined analysis when $\lambda_0$ does not correspond to a locally constant multiplicity has been done by Lannes [107].
- The argument developed in the proof of Theorem 1.6 can be used in the context of parabolic-hyperbolic operators. We leave the reader to prove the following result (**Hint**: show that, for $\xi$ large enough, the appropriate matrix has an invariant subspace $N(\xi)$, which tends to the subspace defined by $v=0$ as $\xi \to +\infty$).

**Theorem 1.8** *Assume that the Cauchy problem for the system*

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u + \sum_\alpha B^\alpha \partial_\alpha v = 0,$$
$$\partial_t v + \sum_\alpha C^\alpha \partial_\alpha u + \sum_\alpha D^\alpha \partial_\alpha v = \sum_{\alpha,\beta} E^{\alpha\beta} \partial_\alpha \partial_\beta v$$

*is well-posed in $L^2(\mathbb{R}^d \times \mathbb{R}_t^+)$. Assume also that the diffusion matrix*

$$E(\xi) := \sum_{\alpha,\beta} \xi_\alpha \xi_\beta E^{\alpha\beta}$$

*is non-singular for every $\xi \neq 0$. Then the operator*

$$\partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

*is hyperbolic.*

- Likewise, one can consider first-order systems with damping (see [19, 147, 222, 223]). Again, we leave the reader to prove the following result (**Hint**: for $\xi=0$, the subpace defined by $v=0$ is invariant for the appropriate matrix. Extend it as an invariant subspace $N(\xi)$.)

**Theorem 1.9** *Let $R \in \mathbf{M}_p(\mathbb{R})$ be given, with $1 \le p < n$. Assume that the Cauchy problem for the system*

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u + \sum_\alpha B^\alpha \partial_\alpha v = 0,$$

$$\partial_t v + \sum_\alpha C^\alpha \partial_\alpha u + \sum_\alpha D^\alpha \partial_\alpha v = Rv$$

*is well-posed in $L^2(\mathbb{R}^d \times \mathbb{R}_t^+)$, uniformly in time, in the sense that there exists a constant $M$, independent of time, such that every solution satisfies*

$$\|(u,v)(t)\|_{L^2} \le M \|(u,v)(0)\|_{L^2}.$$

*Assume also that the damping matrix $R$ is non-singular. Then the operator*

$$\partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

*is hyperbolic.*

This result is meaningful in the study of *relaxation models.*

### 1.5.2 *Strichartz estimates*

This section deals with norms of $L_t^p(L_x^q)$ type for functions $u(x,t)$, namely

$$\|u\|_{p,q} := \left( \int_{\mathbb{R}} \|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^p dt \right)^{1/p}.$$

Such norms define Banach spaces. Interpolation between the spaces associated to pairs $(p_1, q_1)$ and $(p_2, q_2)$ (say $p_1 \le p_2$) yields the spaces associated to $(p,q)$, with

$$p_1 \le p \le p_2, \quad \left( \frac{1}{q_1} - \frac{1}{q_2} \right) \left( \frac{1}{p} - \frac{1}{p_2} \right) = \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \left( \frac{1}{q} - \frac{1}{q_2} \right).$$

There are various types of Strichartz estimates. We shall neither list them all, nor give proofs, except in a single particular case (see below). Given a hyperbolic operator $L = \partial_t + A(\nabla_x)$, a Strichartz estimate is an inequality that typically bounds the $L_t^p(L_x^q)$-norm of the solution $u$ of

$$Lu = f, \quad u(t=0) = u_0,$$

in terms of norms of $f$ and $u_0$, taken in other functional spaces. By a duality argument, one deduces the general case from the simpler one $u_0 \equiv 0$. The latter follows from a dispersion inequality in the homogeneous case $f \equiv 0$, through the Fractional Integration Theorem (Hardy–Littlewood–Sobolev inequality).

As far as we know, the wave operator and its variant are the only ones that retained the attention of authors within hyperbolic problems. Therefore, we shall restrict ourselves to operators $L$ that ‘divide’ the Dalembertian $\partial_t^2 - \Delta_x$, in the sense that $A(\xi)^2 \equiv |\xi|^2 I_n$. We then speak of *Dirac operators*, and $Lu = 0$ implies $\partial_t^2 u_j - \Delta_x u_j = 0$ for $j = 1, \dots, n$. The simplest example is

$$\partial_t u + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x u + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y u = 0. \tag{1.5.36}$$

A more complicated one may be built from Pauli matrices:

$$A(\xi) = \begin{pmatrix} \xi_1 I_4 & \xi_2\sigma_2 + \xi_3\sigma_3 + \xi_4\sigma_4 \\ \xi_2\sigma_2^T + \xi_3\sigma_3^T + \xi_4\sigma_4^T & -\xi_1 I_4 \end{pmatrix},$$

with

$$\sigma_2 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0_2 & J \\ -J & 0_2 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the seminal work by Strichartz [200], the homogeneous case is treated by noting that the Fourier transform of $u$ is supported by the characteristic cone. It is important that, away from its singularity, this cone have non-zero curvature. In particular, we do not expect Strichartz estimates to hold when $\mathrm{char}(L)$ has a flat component, a fact that happens in linearized gas dynamics for instance, or in one-space dimension. In subsequent studies (see [96,203]), the estimate is obtained as a consequence of the conservation of energy, a dispersion inequality (an algebraic decay of $\|u(t)\|_{L^q}$ when $f \equiv 0$), Hardy–Littlewood–Sobolev inequalities and a so-called $T^*T$ argument.

A typical Strichartz inequality for the wave equation $\partial_t^2\phi - \Delta_x\phi = 0$ is

$$\|\phi\|_{L_t^p(L_x^q)} \leq c(p,q,d)\|\nabla_{x,t}\phi|_{t=0}\|_{L^2}, \tag{1.5.37}$$

which holds when

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - 1, \quad \frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}, \quad 2 \leq p, q \leq \infty, \quad d \geq 2, \tag{1.5.38}$$

with the exception of the triplet $(p,q,d) = (2,\infty,3)$. Translating in terms of $u$, we obtain an inequality

$$\|u\|_{L_t^p(L_x^q)} \leq c(p,q,d)\|u_0\|_{\dot{H}^1}, \tag{1.5.39}$$

where $\dot{H}^1(\mathbb{R}^d)$ denotes the homogeneous Sobolev space of tempered distributions such that $\xi\hat{u}(\xi)$ is square-integrable. If $d \geq 3$, (1.5.39) contains the endpoint case

$$\|u\|_{L_t^\infty(L_x^{2^*})} \leq c(d)\|u_0\|_{\dot{H}^1},$$

where

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$$

is the Sobolev exponent:

$$\dot{H}^1(\mathbb{R}^d) \subset L^{2^*}(\mathbb{R}^d).$$

This particular inequality is an obvious consequence of the Sobolev embedding and the constancy of $\|u(t)\|_{\dot H^1}$.

For the same trivial reason, if $s \in (0, d/2)$, a 'Strichartz inequality' holds of the form

$$\|u\|_{L_t^\infty(L_x^{q(s)})} \le c(s,d)\|u_0\|_{\dot H^s}, \tag{1.5.40}$$

where

$$\frac{1}{q(s)} = \frac{1}{2} - \frac{s}{d}.$$

Again, as in (1.5.39), this trivial result is simply the endpoint of a list of non-trivial ones. We also have

$$\|u\|_{L_t^p(L_x^q)} \le c(p,q,s,d)\|u_0\|_{\dot H^s} \tag{1.5.41}$$

for

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s, \quad \frac{2}{p} + \frac{d-1}{q} \le \frac{d-1}{2}, \quad 2 \le p, \quad d \ge 2, \tag{1.5.42}$$

with the exception of the triplets

$$(p,q,s) = \left(\frac{4}{d-1}, \infty, \frac{d+1}{4}\right).$$

We emphasize that (1.5.41) is scale invariant, in the sense that both sides have the same degree of homogeneity when $u$ is replaced by $u^\lambda$, where $u^\lambda(x,t) := u(\lambda x, \lambda t)$, another solution of $Lv = 0$.

**Strichartz estimates vs $L^p$-well-posedness** From Brenner's theorem [22, 23], the Cauchy problem for a Dirac operator is ill-posed in every $L^p$-space but $L^2$. As a matter of fact, the matrices $A^\alpha$ of a Dirac operator satisfy

$$(A^\alpha)^2 = I_n, \quad A^\alpha A^\beta + A^\beta A^\alpha = 0_n \quad (\alpha \ne \beta),$$

which immediately imply $[A^\alpha, A^\beta] \ne 0_n$. We shall see that the ill-posedness may be viewed as a consequence of Strichartz estimates. This must be a rather general fact, as the lack of commutation of the matrices $A^\alpha$ of a hyperbolic operator $L$, is needed in order that $\mathrm{char}(L)$ have non-zero curvature.

Let $P$ be the set of exponents $p$ such that the Cauchy problem for $L$ is well-posed in $L^p$. Obviously, 2 belongs to $P$. By standard interpolation theory (Riesz–Thorin theorem, see [15]), $P$ is an interval. Next, the fact that $L^*$ is also a Dirac operator, plus a duality argument, show that $P$ is symmetric with respect to the involution $p \mapsto p'$ (as usual, $1/p + 1/p' = 1$).

Assume that $P$ contains some element $q > 2$. Hence the solution operator $S_t$ is uniformly bounded on $L^q$ for $t \in (-1, 1)$. Let $s > 0$ be such that $q > q(s)$, and

$q$ satisfies the inequalities in (1.5.42), so that (1.5.41) applies for some $p$. Writing

$$u(0) = \int_0^1 u(0)\,\mathrm{d}t = \int_0^1 S_{-t}u(t)\,\mathrm{d}t,$$

and using (1.5.41), we obtain an inequality

$$\|w\|_{L^q} \leq c\|w\|_{\dot{H}^s}, \quad \forall w \in \dot{H}^s(\mathbb{R}^d).$$

Such an inequality is obviously false, for instance because it is not scale invariant. We deduce that

$$P \subset [1, 2].$$

Since $P$ is symmetric upon $p \mapsto p'$, we conclude that $P = \{2\}$, confirming Brenner's theorem for Dirac operators.

**A proof for the 3-dimensional wave equation** We give here the proof of (1.5.37) for the wave equation

$$\partial_t^2 \phi = \Delta_x \phi \tag{1.5.43}$$

in the special case $d = 3$. The constraints on $(p, q)$ are therefore

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \quad 2 < p \leq \infty \quad (\text{that is } 6 \leq q < \infty).$$

To keep the presentation as short as possible, we limit ourselves to the proof of the inequality when $\phi|_{t=0} = 0$. We recall that, denoting $\phi_1$ the time derivative of $\phi$ at initial time, the solution of (1.5.43) is given by

$$\phi(x, t) = \frac{1}{4\pi t} \int_{S(x;t)} \phi_1(y)\,\mathrm{d}s(y),$$

where $S(x; t)$ is the sphere of radius $t$, centred at $x$, and $ds(y)$ is the area element. Denote $P_t$ the operator $\phi_1 \mapsto \phi(t)$. Fourier transforming the wave equation, we have easily

$$\widehat{P_t\chi}(\xi) = \frac{\sin t|\xi|}{|\xi|}\hat{\chi}(\xi),$$

which justifies the notation

$$P_t = \frac{\sin t|D|}{|D|}.$$

Since the symbol of $P_t$ is real, it is a self-adjoint operator. The operator $P_s^* P_t = P_s P_t$ has symbol

$$\frac{(\sin t|\xi|)\,(\sin s|\xi|)}{|\xi|^2} = \frac{\cos(t-s)|\xi| - \cos(t+s)|\xi|}{2|\xi|^2}.$$

Therefore $P_s^* P_t$ is the convolution operator of kernel $(K(\cdot, t-s) - K(\cdot, t+s))/2$, where

$$K(\cdot, t) := \mathcal{F}^{-1} \frac{\cos t|\xi|}{|\xi|^2}.$$

It is not too difficult to compute, for $t > 0$,

$$K(x; t) = \frac{H(|x| - t)}{4\pi |x|},$$

where $H$ is the Heaviside function. It follows immediately that

$$\|K(\cdot, t)\|_{L^r(\mathbb{R}^3)} = c_r t^{-1+3/r}, \quad r > 3.$$

From Young's inequality, we deduce (take $r = q/2$)

$$\|P_s^* P_t f\|_{L^q} \leq c(q) \left( |t-s|^{-1+6/q} + |t+s|^{-1+6/q} \right) \|f\|_{L^{q'}}, \tag{1.5.44}$$

provided that $q > 6$.

Given a function $f \in L^{p'}(0, +\infty; L_x^{q'})$, let us form

$$v := \int_0^\infty P_t f(t) \, \mathrm{d}t.$$

The $T^*T$ argument consists in estimating $v$ in $L^2(\mathbb{R}^3)$. First, we have

$$\begin{aligned} \|v\|_{L^2}^2 &= \left\langle \int_0^\infty P_t f(t) \, \mathrm{d}t, \int_0^\infty P_s f(s) \, \mathrm{d}s \right\rangle \\ &= \int_0^\infty \int_0^\infty \mathrm{d}t \, \mathrm{d}s \, \langle P_s^* P_t f(t), f(s) \rangle. \end{aligned}$$

Using (1.5.44) and the fact that $|t-s| \leq |t+s|$, we obtain

$$\begin{aligned} \|v\|_{L^2}^2 &\leq \int_0^\infty \int_0^\infty \mathrm{d}t \, \mathrm{d}s \, \|P_s^* P_t f(t)\|_{L^q} \|f(s)\|_{L^{q'}} \\ &\leq 2c(q) \int_0^\infty \int_0^\infty |t-s|^{-1+6/q} \|f(t)\|_{L^{q'}} \|f(s)\|_{L^{q'}} \mathrm{d}t \, \mathrm{d}s. \end{aligned}$$

Defining

$$G(t) := \int_0^\infty |t-s|^{-1+6/q} \|f(s)\|_{L^{q'}} \mathrm{d}s,$$

we infer from the Hölder inequality

$$\|v\|_{L^2}^2 \leq 2c(q) \, \|G\|_{L^p(0,+\infty)} \|f\|_{L_t^{p'}(L_x^{q'})}.$$

From the Hardy–Littlewood–Sobolev inequality, we also know

$$\|G\|_{L^p(0,+\infty)} \leq \gamma(q) \, \|f\|_{L_t^{p'}(L_x^{q'})},$$

provided that

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}.$$

It follows that

$$\|v\|_{L^2}^2 \leq c'(q)\,\|f\|_{L_t^{p'}(L_x^{q'})}^2. \tag{1.5.45}$$

We conclude via a duality argument. First:

$$\begin{aligned}\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^3} f\phi \mathrm{d}x &= \int_0^{+\infty} \langle P_t\phi_1, f(t)\rangle \,\mathrm{d}t \\ &= \langle \phi_1, \int_0^{+\infty} P_t f(t)\,\mathrm{d}t\rangle.\end{aligned}$$

Using (1.5.45), we deduce

$$\left|\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^3} f\phi \mathrm{d}x\right| \leq \sqrt{c'(q)}\,\|\phi_1\|_{L^2}\|f\|_{L_t^{p'}(L_x^{q'})}.$$

Therefore,

$$\|\phi\|_{L_t^p(L_x^q)} = \sup_f \frac{\left|\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^3} f\phi \mathrm{d}x\right|}{\|f\|_{L_t^{p'}(L_x^{q'})}} \leq \sqrt{c'(q)}\,\|\phi_1\|_{L^2},$$

which is precisely the expected inequality in our case.

### 1.5.3 *Systems with differential constraints*

Several examples in natural sciences involve systems of a slightly more general form than (1.0.1), because of differential constraints that are satisfied by $u(t)$ at every time interval. Let us consider the homogeneous case, with $B = 0$. Then a typical system has the form

$$\partial_t u + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha u = 0, \quad \sum_{\beta=1}^{d} C^\beta \partial_\beta u = 0, \tag{1.5.46}$$

where $C^\beta \in M_{p\times n}(\mathbb{R})$, $p$ being the number of constraints.

Such systems occur whenever one rewrites a higher-order system as a first-order one. Let us take, as an example, the wave equation

$$\partial_t^2 \phi = c^2 \Delta\phi, \tag{1.5.47}$$

where $c > 0$ is the wave velocity. Every solution of (1.5.47) yields a solution $u := (c\nabla_x\phi, -\partial_t\phi)$ of (1.2.12), where

$$A(\xi) = \begin{pmatrix} 0_d & c\xi \\ c\xi^T & 0 \end{pmatrix}.$$

Since $A(\xi)$ is symmetric, the corresponding system is hyperbolic. The spectrum of $A(\xi)$ is easily computed and consists of the simple eigenvalues $\pm c|\xi|$, and the multiple eigenvalue 0. The latter is actually spurious, only due to the fact that the mapping $\phi \mapsto u$ is not onto. Therefore, some solutions $u$ do not correspond to solutions of the wave equation, whence the need of the constraint $\mathrm{curl} v = 0$ on the $d$ first components of $u = (v, w)$.

Another example is provided by Maxwell's system, which writes, in the absence of electric charges, as

$$\begin{aligned} \partial_t B + \mathrm{curl} D &= 0, \\ \partial_t D - \mathrm{curl} B &= 0, \\ \mathrm{div} B &= 0, \\ \mathrm{div} D &= 0. \end{aligned}$$

Again, the evolutionary part (the two first equations above), constitute a symmetric system, therefore a hyperbolic one. We compute easily the eigenvalues, $\pm|\xi|$ and 0, where zero has no physical significance, and must be ruled out with the help of the constraints.

Other examples come from field equations in relativity, where gauge invariance implies that natural variables are redundant.

The general philosophy is that the initial data is given satisfying the constraints, and the evolution must preserve them. Using a Fourier transform, it amounts to saying that $C(\xi)A(\xi)v$ must vanish when $C(\xi)v$ does. In other words, the kernel $N(\xi)$ of $C(\xi)$ is an invariant subspace of $A(\xi)$. As noticed by Dafermos [45], this property is fulfilled as soon as $C^\alpha A^\beta + C^\beta A^\alpha = 0$ holds for every pair $(\alpha, \beta)$. As a matter of fact, these identities, which hold frequently, imply $C(\xi)A(\xi) = 0$. In practice, all examples satisfy the following assumption **(CR)**:

for non-zero vectors $\xi \in \mathbb{R}^d$, the rank of $C(\xi)$ is constant.

This implies that the vector space $N(\xi)$ has a constant dimension and that it depends analytically on $\xi$.

We now characterize strong well-posedness of the Cauchy problem for (1.5.46). Standard functional spaces must be redefined according to the constraint. For instance, $L^2$-well-posedness is concerned with the following space

$$Z := \{u \in L^2(\mathbb{R}^d)^n \,;\, \sum_\beta C^\beta \partial_\beta u = 0\}.$$

Equipped with the usual $L^2$-norm, $Z$ is a Hilbert space. For an initial datum $a \in Z$, the solution is formally given by the formula

$$\hat{u}(\xi, t) = \exp(-itA(\xi))\hat{a}(\xi). \tag{1.5.48}$$

Since $a \in Z$, we know that $\hat{a}(\xi)$ belongs to $N(\xi)$ for almost every $\xi$. Then an estimate of the form $\|u(t)\|_Z \leq C\|a\|_Z$ holds if and only if

$$\sup_\xi \|\exp(-itA_N(\xi))\| < +\infty, \tag{1.5.49}$$

where $A_N(\xi)$ is the restriction of $A(\xi)$ to its invariant subspace $N(\xi)$. As before, (1.5.49) holds for some non-zero time $t_0$ if and only if it holds for every time $t \in \mathbb{R}$. For instance, the choice $t = -1$ gives the criterion for $L^2$-well-posedness:

$$\sup_\xi \| \exp(iA_N(\xi))\| < +\infty. \tag{1.5.50}$$

The $L^2$-well-posedness is again called *hyperbolicity*. As before, it requires that $A_N(\xi)$ (but not necessarily $A(\xi)$) be diagonalizable with real eigenvalues. It may be read in the light of the Kreiss–Strang Theorem 1.2, but for practical purposes, it is useful to consider two classes of well-posed system. The first one consists in the *constantly hyperbolic* systems, namely those for which $A_N(\xi)$ is diagonalizable with real eigenvalues of constant multiplicities, when $\xi \neq 0$. Strict or constant hyperbolicity still implies hyperbolicity.

The second important class consists in the *Friedrichs-symmetrizable* systems. Symmetrizability is the property that there exist a real symmetric definite-positive matrix $S$ and a matrix $M \in M_{n\times p}(\mathbb{R})$, such that $S^\alpha := SA^\alpha + MC^\alpha$ is symmetric, for every $\alpha = 1, \dots, d$. Such systems obey the following energy identity

$$\partial_t(Su, u) + \sum_\alpha \partial_\alpha(S^\alpha u, u) = 0, \tag{1.5.51}$$

which yields the estimate

$$\int_{\mathbb{R}^d} (Su(x,t), u(x,t))\, \mathrm{d}x = \int_{\mathbb{R}^d} (Sa(x), a(x))\, \mathrm{d}x. \tag{1.5.52}$$

The positiveness of matrix $S$ may actually be relaxed in a non-trivial way. For that, let us define a cone $\Lambda$ in $\mathbb{R}^n$, by

$$\Lambda := \{\lambda \in \mathbb{R}^n \,;\, \exists \xi \neq 0,\, C(\xi)\lambda = 0\} = \bigcup_{\xi \neq 0} N(\xi).$$

The following statement is called *compensated compactness*.

**Theorem 1.10** (Murat [145], Tartar [202]) *Let $S$ be a symmetric $n \times n$ matrix. The quadratic form*

$$v \mapsto \int_{\mathbb{R}^d} (Sv, v)\, \mathrm{d}x$$

*is positive-definite on $Z$ if and only if $(S\lambda, \lambda) > 0$ for every non-zero $\lambda \in \Lambda$. In such a case, its square root defines a norm equivalent to $\| \cdot \|_Z$.*

From this, we again obtain an $L^2$ estimate from (1.5.52), in some cases where there does not exist a positive-definite symmetrizer $S$.

**Elastodynamics** An important application of this calculus arises in elastodynamics. Non-linear elastodynamics obeys a second-order system in the unknown $y$ called *displacement*. When written as a first-order system in terms of the first derivatives $u_{j\alpha} := \partial_\alpha y_j$ ($1 \le j \le d$, $0 \le \alpha \le d$ with $\partial_0 := \partial_t$), it must be supplemented with the compatibility relations

$$\partial_\alpha u_{j\beta} - \partial_\beta u_{j\alpha} = 0, \quad \alpha, \beta, j \ge 1.$$

We immediately compute that $v \in \Lambda$ if and only if the submatrix $(v_{j\alpha})_{1\le\alpha,j\le d}$ has rank at most one. For hyperelastic materials, this non-linear system is endowed with an energy density $W(\nabla y)$. A natural restriction is that the map $x \mapsto y$ preserves the orientation, so that $\det \nabla y > 0$ everywhere. In particular, the energy density $W(F)$ must become infinite as $\det F$ tends to zero. Besides, the frame indifference implies that $W(QF) = W(F)$ for every $F, Q$ with $\det F > 0$ and $Q \in SO_d(\mathbb{R})$. It is shown in [36] (Theorem 4.8.1, page 170) that such a function cannot be convex[7]. Now, let us choose a matrix $F$ in the vicinity of which $W$ is not convex, locally. The constant state $\bar{u}$ defined by $\bar{u}_{j\alpha} := F_{j\alpha}$ if $\alpha \ne 0$ and zero otherwise is an equilibrium. Let us linearize the system about $\bar{u}$. The resulting system has constant coefficients and obeys the same differential constraints as the non-linear one. It is compatible with an energy identity (1.5.51), where $(Su, u)$ encodes the second-order terms of the Taylor expansion of the full mechanical energy at $\bar{u}$. In particular, $S$ is not positive. However, $W$ can be *quasiconvex* at $F$, in the sense of Morrey [143], which means

$$\int_{\mathbb{R}^d} W(F + \nabla\psi)\,\mathrm{d}x \ge 0, \quad \forall \psi \in \mathscr{D}(\mathbb{R}^d). \tag{1.5.53}$$

Quasiconvexity implies the *Legendre–Hadamard inequality*

$$(S\lambda, \lambda) \ge 0, \quad \lambda \in \Lambda, \tag{1.5.54}$$

a weaker property than convexity. When (1.5.54) holds strictly for non-zero $\lambda$, the compensated-compactness Theorem tells us that (1.5.52) is a genuine estimate in $Z$. In such a case, the linearized problem is strongly $L^2$-well-posed.

We shall not consider in this chapter the local well-posedness of the non-linear system. The Cauchy problem for quasilinear systems of conservation laws is treated in Chapter 10. Let us mention only that a system of conservation laws endowed with a convex 'entropy' has a well-posedness property within smooth data and solutions (see Theorem 10.1). In elastodynamics, the system governs the evolution of $u = (v, F) = (\partial_t y, \nabla_x y)$. Since the entropy of our system is the energy $\frac{1}{2}|v|^2 + W(F)$, which is not convex, the above-mentioned theorem does not apply. However, Dafermos [46] has found a way to apply it, by rewriting the system of elastodynamics in terms of $u$ and all minors of the matrix $F$. See also Demoulini *et al.* [48]. As a consequence, the local well-posedness is obtained whenever $W$ is *polyconvex*, that is a convex function of $F$ and its minors.

**Electromagnetism** Let us consider Maxwell's equations. The kernel $N(\xi)$ equals $\xi^\perp \times \xi^\perp$, where $\xi^\perp$ is the orthogonal of $\xi$ in the Euclidean space $\mathbb{R}^3$.

[7] We warn the reader that the phase space $\mathbf{GL}_d^+(\mathbb{R})$, made of matrices $F$ with $\det F > 0$, is not a convex set. Thus the convexity of a function is a meaningless notion.

Therefore $\Lambda$ equals $\mathbb{R}^6$, and the symmetrizability has to be understood in the usual sense. In the vacuum, $H = B$ and $E = D$, for appropriate units. The system is already in symmetric form. In a 'linear' material medium, which may be anisotropic, $H, E$ still are linear functions of $B, D$. For linear as well as non-linear media, there is a stored electromagnetic energy density $W(B, D)$, and $E, H$ are given by the following formulæ (see [37])

$$E_j = \frac{\partial W}{\partial D_j}, \quad H_j = \frac{\partial W}{\partial B_j}.$$

In the linear case, $W$ is a quadratic form. The Maxwell system is compatible, as long as we consider $\mathscr{C}^1$ solutions, with the *Poynting identity*, which expresses the conservation of energy

$$\partial_t W(B, D) + \mathrm{curl}(E \times H) = 0. \tag{1.5.55}$$

Let us consider the linearized system about some constant state $(\bar{B}, \bar{D})$. The former considerations show that if the matrix $S := \mathbf{D}^2 W(\bar{B}, \bar{D})$ is positive-definite, then the linear Cauchy problem is $L^2$-well-posed. We can actually relax the convexity condition, with the following observation. The Maxwell system is also compatible with the extra conservation law (herebelow, $u := (B, D)$)

$$\partial_t(B \times D) + \mathrm{div}\left(\frac{\partial W}{\partial B} \otimes B\right) + \mathrm{div}\left(\frac{\partial W}{\partial D} \otimes D\right) + \nabla(W - u \cdot \nabla_u W) = 0.$$

At the linearized level, we may consider a modified energy density ${}^t u S u + \det(X, B, D)$, where $X$ is a given vector in $\mathbb{R}^3$. If there exists an $X$ such that ${}^t u S u + \det(X, B, D)$ is positive-definite, then the linear system is Friedrichs symmetrizable and the Cauchy problem is $L^2$-well-posed. An obvious necessary condition for such an $X$ to exist is that ${}^t u S u > 0$ whenever $B \times D = 0$ and $(B, D) \neq 0$. At the non-linear level, the same procedure as the one imagined by Dafermos in elastodynamics may be employed. The result is that the non-linear Maxwell's system is locally well-posed for smooth initial data and smooth solutions, whenever $W$ can be written as a convex function of $B$, $D$ and $B \times D$. (See [21, 188].)

### 1.5.4 *Splitting of the characteristic polynomial*

We give in this section a property of the characteristic polynomial $(X; \xi) \mapsto \det(X I_n + A(\xi))$, when the operator $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is constantly hyperbolic.

Let us begin with an abstract result.

**Lemma 1.3** *Let $P(X; \theta_1, \cdots, \theta_d)$ be a homogeneous polynomial of degree $n$ in $1 + d$ variables, with real coefficients. Assume that the coefficient of $X^n$ is non-zero. Assume also that for all $\theta$ in a non-void open subset $\mathcal{O}$ of $\mathbb{R}^d$, the polynomial $P_\theta := P(\cdot, \theta)$ has a root with multiplicity $\geq 2$. Then $P$ is reducible in $\mathbb{R}[X, \theta]$.*

**Proof** Let us denote by $R := \mathbb{R}[\theta_1, \cdots, \theta_d]$ the factorial ring of polynomials in $d$ variables $\theta$ and by $k := \mathbb{R}(\theta_1, \cdots, \theta_d)$ the field of rational fractions in $\theta$. We first consider $P$ as an element of $k[X]$. Let us recall that $k[X]$ is a Euclidean ring, which has therefore a greatest common divisor (g.c.d.)

Let $Q$ be the g.c.d of $P$ and $P'$ in $k[X]$, a monic polynomial of $X$. Its coefficients, belonging to $k$, are rational fractions of $\theta$. We denote by $Z$ the zero set of the product of denominators of these fractions; $Z$ is a closed set with empty interior.

When $\theta \in \mathcal{O} \setminus Z$ (this is a non-void open set), $Q_\theta := Q(\cdot, \theta)$ has a non-trivial root, which means that either $Q_\theta \equiv 0$ or $d^\circ Q_\theta \geq 1$. However, the condition $Q_\theta \equiv 0$ defines a non-trivial algebraic manifold $M$ (the intersection of the zero sets of the coefficients of $Q$), again a closed set with empty interior. Therefore, there exists a $\theta$ for which $d^\circ Q_\theta \geq 1$, and consequently $d^\circ_X Q \geq 1$.

Since $Q$ divides $P$ in $k[X]$, we write $P = QT$, with $T \in k[X]$. Multiplying by the l.c.m. of the denominators of all coefficients of $Q$ and $T$ (a least common multiple (l.c.m.) and a g.c.d. do exist in the factorial ring $R$), we have $g(\theta)P = Q_1 T_1$, where $g \in A$, $Q_1, T_1 \in R[X]$ and $0 < d^\circ_X Q_1 < n$. We recall that the *contents* of a polynomial $S \in R[X]$, denoted by $c(S)$, is the g.c.d. of all its coefficients. From Gauss' Lemma, $c(Q_1 T_1) = c(Q_1)c(T_1)$ and therefore $g = c(Q_1)c(T_1)$, since $c(P) = 1$ by assumption. We conclude that $P = Q_2 T_2$, where $Q_2 := c(Q_1)^{-1} Q_1 \in R[X]$ and similarly $R_2 \in R[X]$. Moreover, $0 < d^\circ_X Q_2 < n$, which shows that $P$ is reducible in $R[X] = \mathbb{R}[X, \theta]$. □

**Corollary 1.2** *Let $P \in \mathbb{R}(X, \theta)$ be homogeneous with $d^\circ_X P = d^\circ_{(X,\theta)} P$. Let*

$$P = \prod_{l=1}^{L} P_l^{q_l}$$

*its factorization into irreducible factors in $\mathbb{R}(X, \theta)$, the $P_l$s being pairwise distinct.*

*Then each $P_l$ has the following property: for an open dense subset of values of $\theta$ in $\mathbb{R}^d$, the roots of $P_l(\cdot, \theta)$ are simple.*

We now apply the corollary to the characteristic polynomial.

**Proposition 1.7** *Let the operator $L := \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be constantly hyperbolic. Then the characteristic polynomial $\det(XI_n + A(\xi))$ splits as a product*

$$\prod_{l=1}^{L} P_l^{q_l}, \tag{1.5.56}$$

*where the $P_l$s, normalized by $P_l(1, 0) = 1$, satisfy*

- *Each $P_l$ is a homogeneous polynomial of $(X; \xi)$,*
- *The $P_l$s are irreducible, pairwise distinct,*
- *For $\xi \in \mathbb{R}^d \setminus \{0\}$, the roots of $P_l(\cdot, \xi)$ are real and simple,*
- *For $\xi \in \mathbb{R}^d \setminus \{0\}$ and $l \neq k$, $P_l(\cdot, \xi)$ and $P_k(\cdot, \xi)$ do not have a common root.*

**An example** Let us consider Maxwell's equations, where $d = 3$, $n = 6$ and

$$A(\xi) = \begin{pmatrix} 0_3 & J(\xi) \\ -J(\xi) & 0_3 \end{pmatrix}, \quad J(\xi)V := \xi \times V.$$

An elementary computation gives:

$$\det(XI_6 + A(\xi)) = X^2(X^2 - |\xi|^2)^2.$$

Hence the splitting described in Proposition 1.7 corresponds to $P_1(X, \xi) = X$, $P_2(X, \xi) = X^2 - |\xi|^2$, $q_1 = q_2 = 2$.

### 1.5.5 *Dimensional restrictions for strictly hyperbolic systems*

We begin with a matrix theorem, due to Lax [111] in the case $n \equiv 2 \pmod 4$, and to Friedland *et al.*, [62] in the case $n \equiv 3, 4, 5 \pmod 8$:

**Theorem 1.11** *Assume that* $n \equiv 2, 3, 4, 5, 6 \pmod 8$. *Let* $V$ *be a subspace of* $\mathbf{M}_n(\mathbb{R})$ *with the property that every non-zero element in* $V$ *has its eigenvalues real and pairwise distinct. Then* $\dim V \leq 2$.

In terms of hyperbolic operators, this tells us that strictly hyperbolic operators in space dimension $d \geq 3$ can exist only if $n \equiv 0, \pm 1 \pmod 8$ (assuming that the operator really involves all the space variables). This explains why constantly hyperbolic operators occur so frequently, as they exist in space dimension three at every size $n \geq 4$. The simplest examples are:

$n = 4$. Linearized isentropic gas dynamics.

$n = 5$. Linearized non-isentropic gas dynamics.

$n = 6$. Maxwell's equations. We know also of a non-equivalent example of this size.

**Proof** We prove only the Lax case $n \equiv 2 \pmod 4$. We argue by contradiction, assuming that $\dim V \geq 3$. We label the eigenvalues in the increasing order:

$$\lambda_1(M) < \cdots < \lambda_n(M), \quad M \neq 0_n.$$

Every non-zero element $M$ in $V$, having real and distinct eigenvalues, is associated with finitely many (precisely $2^n$) unitary bases of $\mathbb{R}^n$, which depend continuously on $M$. In other words, the set of (real) unitary eigenbases is a finite covering (with $2^n$ sheets) of $V \setminus \{0_n\}$. Since the base space is simply connected (because $d \geq 3$), the covering is trivial and a continuous map $M \mapsto \mathcal{B}(M)$ can actually be defined globally, where

$$\mathcal{B}(M) = \{r_1(M), \ldots, r_n(M)\}$$

is an eigenbasis. By continuity, all the bases $\mathcal{B}(M)$ have the same orientation.

On the one hand, it holds that

$$\lambda_j(-M) = -\lambda_{n-j+1}(M), \quad j = 1, \ldots, n.$$

It follows that

$$r_j(-M) = \pm r_{n-j+1}(M).$$

By continuity, the sign $\pm$ above is constant and depends only on $j$, but not on $M$. Denote it $\rho_j$. Exchanging $j$ with $n-j+1$, we obtain

$$\rho_{n-j+1}\rho_j = 1. \tag{1.5.57}$$

On the other hand, we know that $\{r_1(M), \dots, r_n(M)\}$ and $\{\rho_1 r_n(M), \dots, \rho_n r_1(M)\} = \{r_1(-M), \dots, r_n(-M)\}$ have the same orientation. Since $n \equiv 2 \pmod 4$, the order reversal[8]

$$\{r_1(M), \dots, r_n(M)\} \mapsto \{r_n(M), \dots, r_1(M)\}$$

reverses the orientation. Therefore, it must hold that

$$\prod_{j=1}^{n} \rho_j = -1.$$

This, however, is incompatible with (1.5.57) when $n$ is even. □

Note that if $\dim V = 2$ (that is for strictly hyperbolic operators in two space dimensions), there does not need to exist a continuously defined eigenbasis on $V \setminus \{0_n\}$, as this set is not simply connected. For instance, the system (1.5.36) does not have this property: When following a loop around the origin in $V$, the eigenvectors are flipped.

### 1.5.6 *Realization of hyperbolic polynomial*

Let $p(X_0, \dots, X_d)$, a homogeneous polynomial of degree $n$, be *hyperbolic* with respect to a vector $T \in \mathbb{R}^{d+1}$ in the sense of Gårding (see Section 1.4.4).

Given a hyperbolic polynomial $p$ of degree $n$ in $d+1$ variables, one may always assume that $T$ is the first element $\vec{e}^{\,0}$ of the canonical basis. A natural question is whether $p$ can be realized as $p_L$ for some hyperbolic operator $L$. The case $d = 1$ is easy. Lax [110] conjectured that if $d = 2$, the answer is positive and one can choose a Friedrichs-symmetrizable operator. This has been proved recently by Lewis *et al.* [114], following a result by Helton and Vinnikov [82]. One easily sees that the hyperbolic polynomial $q(X) := X_0^2 - X_1^2 - \cdots - X_d^2$ cannot be realized if $d \geq 3$. However, it may happen that some power $q^\ell$ be realizable, as in the case of Maxwell's system, or Dirac systems. The fundamental question whether every hyperbolicity cone can be realized as a forward cone for some hyperbolic operator remains open so far.

Notice that two different hyperbolic operators $L$ and $L'$ can yield the same polynomial,

$$p_L = p_{L'}.$$

[8] This part of the proof would also work when $n \equiv 3 \pmod 4$.

This happens at least when $L'$ is obtained from $L$ by a linear change of variables. Then $(A^\alpha)' = P^{-1}A^\alpha P$ for some non-singular matrix. We can also make only a linear combination of the equations, which yields an operator $L'' = S_0\partial_t + \sum_\alpha S^\alpha\partial_\alpha$, where $S^\alpha := S_0A^\alpha$, which modifies $p_L$ by a constant factor. Change of co-ordinates should also be allowed.

It is an important problem to classify the hyperbolic operators up to such a change. The characteristic cone is of course an invariant of this problem, but it is not the only one. There actually exist non-equivalent operators that have the same characteristic cone. A way to go forward, which has not been pushed so far, is to consider the characteristic bundle, whose basis is the characteristic variety (the projective set associated to the characteristic cone) and the fibres are the corresponding eigenfields. This bundle is modified by the change of variables and the combinations of equations, by its topology is not. Thus the Chern class of the bundle is a more accurate invariant. When the characteristics have variable multiplicities, the cone and the bundle are not smooth and the analysis becomes more difficult.

A similar problem, perhaps even more important is to classify within the set of symmetrizable operator, since the physics usually provides a Friedrichs symmetrizer, through an energy estimate or an entropy principle. Of course, the characteristic bundle remains a crucial tool. But some other invariants may appear, in particular in the case where $L$ admits a linear velocity

$$\lambda(\xi) = V \cdot \xi,$$

which can be brought to the case $\lambda(\xi) \equiv 0$. See the discussion in Section 6.1.2.