

# 3

## Quadratic functions

This chapter will show you how to

- sketch quadratic graphs
- use different methods to solve quadratic equations
- understand and interpret the discriminant.

### Before you start

You should know how to:

1 Simplify surd forms.

e.g. Simplify  $\sqrt{75}$

Use the rule  $\sqrt{ab} = \sqrt{a}\sqrt{b}$ :

$$\begin{aligned}\sqrt{75} &= \sqrt{25 \times 3} \\ &= \sqrt{25}\sqrt{3} \\ &= 5\sqrt{3}\end{aligned}$$

2 Substitute values into an equation.

e.g. If  $y = x^2 - 3x + 2$

find the value of  $y$  when

i  $x = 3$     ii  $x = -2$

$$\begin{aligned}\text{i } y &= (3)^2 - 3(3) + 2 \\ &= 9 - 9 + 2 \\ &= 2\end{aligned}$$

$$\begin{aligned}\text{ii } y &= (-2)^2 - 3(-2) + 2 \\ &= 4 + 6 + 2 \\ &= 12\end{aligned}$$

3 Expand double brackets.

e.g. Expand  $(x - 3)(2x + 5)$

$$(x - 3)(2x + 5) = 2x^2 + 5x - 6x - 15$$

Simplify by collecting like terms:

$$(x - 3)(2x + 5) = 2x^2 - x - 15$$

Check in:

1 Simplify

a  $\sqrt{90}$

b  $\sqrt{60}$

c  $\sqrt{150}$

d  $\sqrt{8}$

e  $\sqrt{54}$

2 Find  $y$  when

i  $x = 2$  and    ii  $x = -3$

a  $y = x^2 + 2x - 3$

b  $y = 2x^2 - 3x$

c  $y = 2 - x - x^2$

d  $y = 3(2x - 3)^2$

e  $y = (5 - x)(2 - x)$

f  $y = (x + 3)^2 - (x - 3)^2$

3 Expand

a  $(x + 2)(x + 5)$

b  $(2x - 1)(2x - 3)$

c  $(1 - x)(1 - 3x)$

d  $(2x + 1)(1 - 2x)$

e  $(5 - 2x)(2x - 5)$

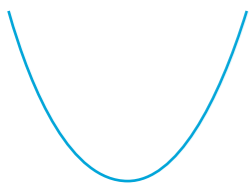
### 3.1 Factorising quadratic equations

The general form of a **quadratic function** is

$$f(x) = ax^2 + bx + c$$

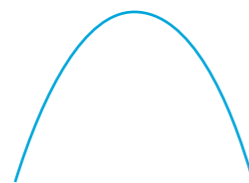
where  $a, b$  and  $c$  are constant values and  $a \neq 0$ .

When  $a > 0$  the graph of  $f(x)$  looks like



The lowest point on the curve is called the **minimum point** (plural **minima**).

When  $a < 0$  the graph of  $f(x)$  looks like



The highest point on the curve is called the **maximum point** (plural **maxima**).

The general form of a **quadratic equation** is

$$ax^2 + bx + c = 0$$

Many quadratic equations can be solved by factorising.

**EXAMPLE 1** Solve the equation  $x^2 - 6x + 8 = 0$

**Factorise:**  $(x - 4)(x - 2) = 0$

Either  $x - 4 = 0$  or  $x - 2 = 0$   
 $x = 4$   $x = 2$

The quadratic equation has two solutions  $x = 2$  or  $x = 4$ .

- A quadratic equation usually has two solutions, but may have only one or none at all.

**EXAMPLE 2** Find  $x$  when  $(x + 4)(x - 1) = 0$

Either  $x + 4 = 0$  or  $x - 1 = 0$

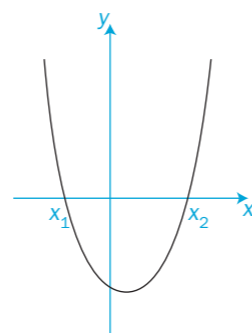
giving  $x = -4$  or  $x = 1$

The solutions are  $x = -4$  or  $x = 1$ .

You may find it useful to sketch the curve.

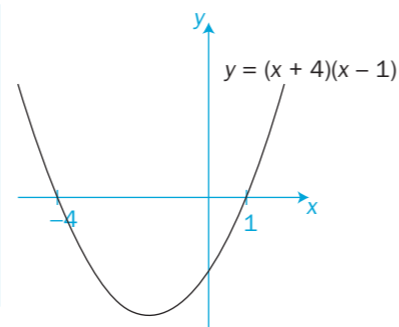
$f(x) = 2x^2 - 3x + 7$  is a quadratic function.

The simplest form of a quadratic function occurs when  $a = 1$ ,  $b = 0$  and  $c = 0$  giving  $f(x) = x^2$



The solutions,  $x_1$  and  $x_2$ , or **roots**, are the values of  $x$  at the  $x$ -intercepts.

If two terms  $A$  and  $B$  multiply to give zero, then either  $A = 0$  or  $B = 0$  or  $A = B = 0$



- A quadratic equation that can be written in the form  $(x + p)(x + q) = 0$  has solutions  $x = -p$  or  $x = -q$ .

This method only works if the quadratic expression equals zero. You may need to rearrange the equation before solving it.

**EXAMPLE 3** Find the values of  $x$  that satisfy  $x(2x + 1) = 4(2x + 1)$

**Rearrange:**  $x(2x + 1) - 4(2x + 1) = 0$   
**Factorise:**  $(2x + 1)(x - 4) = 0$

Either  $2x + 1 = 0$  or  $x - 4 = 0$   
 $2x = -1$   $x = 4$

$$x = -\frac{1}{2}$$

The solutions are  $x = -\frac{1}{2}$  or  $x = 4$ .

Sometimes you will need to factorise the equation before you can solve it.

**EXAMPLE 4** Solve **a**  $x^2 + 9x + 20 = 0$  **b**  $2x^2 - x - 6 = 0$

**a**  $x^2 + 9x + 20 = 0$

$$(x + 4)(x + 5) = 0$$

Either  $x + 4 = 0$  or  $x + 5 = 0$   
 $x = -4$   $x = -5$

The solutions are  $x = -4$  and  $x = -5$ .

**b**  $2x^2 - x - 6 = 0$

$$(2x + 3)(x - 2) = 0$$

**Solve:** Either  $2x + 3 = 0$  or  $x - 2 = 0$   
 $2x = -3$   $x = 2$

$$x = -\frac{3}{2}$$

The solutions are  $x = -\frac{3}{2}$  or  $x = 2$ .

It is often helpful to recognise an expression that is the **difference of two squares (DOTS)**.

$$X^2 - Y^2 = (X + Y)(X - Y)$$

If  $p$  is a solution, you can say that ' $p$  satisfies the equation'.

Notice that  $(2x + 1)$  is common to both terms.

If you need to practise factorising quadratics, look at Chapter 0 on the CD-ROM.

One factor will start with  $2x$  and one with  $x$ .  
 One factor will have a  $+$  and one a  $-$ . This is because the constant is negative.

See Chapter 0 on the CD-ROM.

EXAMPLE 5

- a** Factorise  $4x^2 - 25$   
**b** Hence solve  $4x^2 - 25 = 0$

**a** Rewrite  $4x^2 - 25$  in the form  $X^2 - Y^2$ :

$$4x^2 - 25 = (2x)^2 - (5)^2$$

Apply  $X^2 - Y^2 = (X + Y)(X - Y)$ :

$$4x^2 - 25 = (2x)^2 - (5)^2 = (2x + 5)(2x - 5)$$

**b**  $4x^2 - 25 = 0$

$$\text{so } (2x + 5)(2x - 5) = 0$$

$$\begin{array}{l} \text{Either } 2x + 5 = 0 \quad \text{or} \quad 2x - 5 = 0 \\ \qquad 2x = -5 \qquad \qquad \qquad 2x = 5 \\ \qquad x = -\frac{5}{2} \qquad \qquad \qquad x = \frac{5}{2} \end{array}$$

The solutions are  $x = -\frac{5}{2}$  or  $x = \frac{5}{2}$ .

**Exercise 3.1**

**1** Solve these equations.

- |                                 |                                  |                                |
|---------------------------------|----------------------------------|--------------------------------|
| <b>a</b> $(x - 2)(x - 3) = 0$   | <b>b</b> $x(x + 1) = 0$          | <b>c</b> $2x(x - 3) = 0$       |
| <b>d</b> $3(x + 4)(x - 5) = 0$  | <b>e</b> $(x + 7)^2 = 0$         | <b>f</b> $5(2 - x)(3 - x) = 0$ |
| <b>g</b> $-4(4 - x)^2 = 0$      | <b>h</b> $(x - 5)(2x + 1) = 0$   | <b>i</b> $x(2x - 3) = 0$       |
| <b>j</b> $4x(1 - 3x) = 0$       | <b>k</b> $(2x - 5)(2x - 3) = 0$  | <b>l</b> $5(3x - 5)^2 = 0$     |
| <b>m</b> $(4x + 2)(2 - 4x) = 0$ | <b>n</b> $3(2x + 5)(5x + 2) = 0$ | <b>o</b> $9(3 - 2x)^2 = 0$     |

**2** Factorise these equations and hence find the values of  $x$  that satisfy each equation.

- |                                    |  |
|------------------------------------|--|
| <b>a</b> $2(x - 1) + x(x - 1) = 0$ | <b>b</b> $3(4x + 1) + x(4x + 1) = 0$       |
| <b>c</b> $7(3x - 1) = 2x(3x - 1)$  | <b>d</b> $x(4 - 3x) = \frac{1}{2}(4 - 3x)$ |

**3** **i** Factorise these equations.  
**ii** Hence find the values of  $x$  that satisfy each equation.  
**iii** Sketch the graph of the associated quadratic function.  
 In your sketch, mark the coordinates of the points where the curve crosses the  $x$ - and  $y$ -axes.

- |                             |                            |                               |
|-----------------------------|----------------------------|-------------------------------|
| <b>a</b> $x^2 + 3x + 2 = 0$ | <b>b</b> $x^2 + x - 6 = 0$ | <b>c</b> $x^2 - 7x + 10 = 0$  |
| <b>d</b> $x^2 - x = 30$     | <b>e</b> $x^2 = 11x - 28$  | <b>f</b> $x^2 = 24 - 2x$      |
| <b>g</b> $x^2 = 5x + 36$    | <b>h</b> $x^2 = 13x - 30$  | <b>i</b> $24 + 25x + x^2 = 0$ |

Let  $y = 0$  and  $x = 0$ .

**4** Factorise these equations using the difference of two squares and solve them to find  $x$ .

- |                                      |                           |                                  |
|--------------------------------------|---------------------------|----------------------------------|
| <b>a</b> $x^2 - 9 = 0$               | <b>b</b> $4x^2 - 1 = 0$   | <b>c</b> $9x^2 - 16 = 0$         |
| <b>d</b> $\frac{x^2}{4} - 1 = 0$     | <b>e</b> $27x^2 - 12 = 0$ | <b>f</b> $5 - 125x^2 = 0$        |
| <b>g</b> $\frac{12x^2}{25} - 48 = 0$ | <b>h</b> $2 - x^2 = 0$    | <b>i</b> $3 - \frac{x^2}{4} = 0$ |

**5** Factorise these equations and solve them to find  $x$ .

- |                              |                                |                               |
|------------------------------|--------------------------------|-------------------------------|
| <b>a</b> $2x^2 - x - 1 = 0$  | <b>b</b> $3x^2 - 7x + 2 = 0$   | <b>c</b> $4x^2 + 4x + 1 = 0$  |
| <b>d</b> $6x^2 + 5x + 1 = 0$ | <b>e</b> $12x^2 + 13x + 3 = 0$ | <b>f</b> $6 - 11x + 5x^2 = 0$ |
| <b>g</b> $12x^2 = 25x - 12$  | <b>h</b> $6x^2 = 25 - 5x$      | <b>i</b> $12 + 7x = 10x^2$    |

**6** Solve these equations to find the values of  $x$ .

- |                                    |                               |                                 |
|------------------------------------|-------------------------------|---------------------------------|
| <b>a</b> $4(x + 2)^2 = 0$          | <b>b</b> $x(1 - 2x) = 0$      | <b>c</b> $(3x + 2)(4x - 5) = 0$ |
| <b>d</b> $5(x - 3) - x(x - 3) = 0$ | <b>e</b> $x(2x - 1) = 2x - 1$ | <b>f</b> $x^2 + 8x + 15 = 0$    |
| <b>g</b> $25x^2 - 36 = 0$          | <b>h</b> $3x^2 - 11x = 4$     | <b>i</b> $30x^2 + 49x + 20 = 0$ |
| <b>j</b> $21x^2 = 55x - 14$        |                               |                                 |

**7** Solve

- |                               |                          |                           |
|-------------------------------|--------------------------|---------------------------|
| <b>a</b> $-2x^2 - 3x - 1 = 0$ | <b>b</b> $x^2 = -3x - 2$ | <b>c</b> $-3x^2 + 7x = 2$ |
|-------------------------------|--------------------------|---------------------------|

**INVESTIGATIONS**

**8** Four equations and their solutions are given:

Equations:	Solutions:
1. $x^2 + 3x + 2 = 0$	$x = -1$ $x = 3$
2. $x^2 - 3x + 2 = 0$	$x = 1$ $x = -3$
3. $2x^2 - x - 3 = 0$	$x = -2$ $x = \frac{3}{2}$
4. $x^2 - 4x + 4 = 0$	$x = 2$ $x = -\frac{3}{2}$

Some solutions may be used more than once.

Match each equation with its solution(s).

**9** The box contains some linear factors:

$$(2x + 3) \quad (x - 2) \quad (2x + 1) \quad (3x + 1) \quad (x + 1) \quad (x + 4) \quad (x - 4) \quad (2x - 3) \quad (2x - 1)$$

Use some of these factors to factorise the quadratics

- |                               |                              |                              |
|-------------------------------|------------------------------|------------------------------|
| <b>a</b> $2x^2 - 5x - 12 = 0$ | <b>b</b> $2x^2 - 3x - 2 = 0$ | <b>c</b> $4x^2 + 4x - 3 = 0$ |
|-------------------------------|------------------------------|------------------------------|

## 3.2 Completing the square

Some quadratic expressions have two identical factors. Such expressions are known as **perfect squares**.

EXAMPLE 1

Solve the equation  $x^2 + 10x + 25 = 9$

$$\begin{aligned}x^2 + 10x + 25 &= 9 \\(x + 5)^2 &= 9\end{aligned}$$

Take square roots of each side:

$$x + 5 = \pm 3$$

Rearrange:  $x = -5 \pm 3$

Either  $x = -5 + 3$  or  $x = -5 - 3$

Hence  $x = -2$  or  $x = -8$

- You can rewrite any quadratic expression in factorised form like this:

$$ax^2 + bx + c \equiv a(x + p)^2 + q$$

Factorising an equation into this form is called **completing the square**.

$x^2 + 10x + 25$  is a perfect square since it factorises into  $(x + 5)^2$ .

Remember to include both solutions of  $x^2 = 9$ .  
 $3^2 = 9$  and  $(-3)^2 = 9$

$\equiv$  means 'is identical to'.

4 is chosen because  $4x + 4x = 8x$

$(x - 1)^2 = x^2 - 2x + 1$   
so you must subtract 1.

EXAMPLE 2

Complete the square for the expression  $x^2 + 8x$

Try  $(x + 4)^2$ :

$$(x + 4)^2 = x^2 + 8x + 16$$

This is close, however you want  $x^2 + 8x$

Subtract 16:  $x^2 + 8x = (x + 4)^2 - 16$

If the coefficient of  $x^2$  is not equal to 1, you can still complete the square.

EXAMPLE 3

Complete the square for the expression  $2x^2 - 4x$

First take out the coefficient of  $x^2$ :

$$2x^2 - 4x = 2(x^2 - 2x)$$

Now complete the square inside the bracket:

$$= 2(x^2 - 2x + 1 - 1)$$

$$= 2[(x - 1)^2 - 1] = 2(x - 1)^2 - 2$$

You can use the method of completing the square to solve quadratic equations.

EXAMPLE 4

Use the method of completing the square to solve

a  $x^2 + 10x + 7 = 0$       b  $2x^2 + 5x - 4 = 0$

a  $x^2 + 10x + 7 = 0$   
Rewrite:  $x^2 + 10x = -7$

The coefficient of  $x$  is 10.

$\frac{1}{2} \times 10 = 5$  so try  $(x + 5)^2$ :  $(x + 5)^2 = x^2 + 10x + 25$

Subtract 25:  $(x + 5)^2 - 25 = x^2 + 10x = -7$

Hence  $(x + 5)^2 - 25 = -7$

Rearrange:  $(x + 5)^2 = -7 + 25$   
 $(x + 5)^2 = 18$

Take square roots:  $x + 5 = \pm\sqrt{18}$

Simplify:  $x + 5 = \pm 3\sqrt{2}$

Rearrange:  $x = -5 \pm 3\sqrt{2}$

Either  $x = -5 + 3\sqrt{2}$  or  $x = -5 - 3\sqrt{2}$

b  $2x^2 + 5x - 4 = 0$

Rewrite:  $2\left(x^2 + \frac{5}{2}x - 2\right) = 0$

Since  $2 \neq 0$ , the expression in the bracket must be equal to 0.

Hence  $x^2 + \frac{5}{2}x - 2 = 0$

Rearrange:  $x^2 + \frac{5}{2}x = 2$

The coefficient of  $x$  is  $\frac{5}{2}$ .

$\frac{1}{2} \times \frac{5}{2} = \frac{5}{4}$  so try  $\left(x + \frac{5}{4}\right)^2$ :  $\left(x + \frac{5}{4}\right)^2 = x^2 + \frac{5}{2}x + \frac{25}{16}$

Subtract  $\frac{25}{16}$ :  $\left(x + \frac{5}{4}\right)^2 - \frac{25}{16} = x^2 + \frac{5}{2}x = 2$

Hence  $\left(x + \frac{5}{4}\right)^2 - \frac{25}{16} = 2$

Rearrange:  $\left(x + \frac{5}{4}\right)^2 = 2 + \frac{25}{16}$

Simplify and take square roots:  $x + \frac{5}{4} = \pm\sqrt{\frac{57}{16}}$

Rearrange:  $x = -\frac{5}{4} \pm \frac{\sqrt{57}}{4}$

Hence  $x = \frac{-5 + \sqrt{57}}{4}$  or  $x = \frac{-5 - \sqrt{57}}{4}$

There is a simple rule that can help you in completing the square:

- If you rewrite  $ax^2 + bx + c$  as  $a(x + p)^2 + q$  then  $p = \frac{b}{2a}$

Remember to include both solutions of  $\sqrt{18}$ .

$$\begin{aligned}\sqrt{18} &= \sqrt{(9 \times 2)} \\ &= \sqrt{(3^2 \times 2)} = 3\sqrt{2}\end{aligned}$$

$$\begin{aligned}\sqrt{\left(2 + \frac{25}{16}\right)} &= \sqrt{\left(\frac{32}{16} + \frac{25}{16}\right)} \\ &= \sqrt{\frac{57}{16}}\end{aligned}$$

Leave your answer in surd form.

## Exercise 3.2

1 Take square roots to find the value of  $x$ .

- a  $(x-3)^2 = 25$       b  $(x+2)^2 = 25$   
 c  $(2x+1)^2 = 1$       d  $(2x-3)^2 = 16$   
 e  $(5+2x)^2 = 1$       f  $(3-4x)^2 = 36$   
 g  $(5x-7)^2 = 49$       h  $\left(\frac{x}{2}+2\right)^2 = 25$   
 i  $(x+4)^2 = 3$       j  $(2x-1)^2 = 5$

2 Complete the square for each expression, writing your final answer in the form  $k(x+a)^2 + b$ .

- a  $x^2 + 4x$       b  $x^2 - 4x$   
 c  $x^2 + 5x$       d  $x^2 - 3x$   
 e  $x^2 - 2x$       f  $2x^2 + 4x$   
 g  $3x^2 - 6x$       h  $5x^2 - 3x$   
 i  $\frac{x^2}{2} + 4x$       j  $\frac{3}{4}x^2 - 2x$

3 Use the method of completing the square to solve these equations, leaving your answers in surd form.

- a  $x^2 + 4x + 1 = 0$   
 b  $x^2 - 4x - 2 = 0$   
 c  $2x^2 + 8x + 3 = 0$   
 d  $2x^2 - 4x + 1 = 0$   
 e  $2x^2 - 5 = 6x$   
 f  $5 + 4x - 2x^2 = 0$

## INVESTIGATIONS

4 Completing the square has a useful purpose when sketching quadratics.

Consider the equation  $y = x^2 + 6x + 5$

- a Express  $y$  in the form  $y = (x+a)^2 + b$   
 b Find the value of  $x$  that gives the *minimum* value for  $y$ .  
 c What is the minimum value of  $y$ ? What is the  $y$ -coordinate of the minimum point?  
 d Hence sketch the curve, showing the intersection with the axes and the minimum point.

5 Match the following equations with their minimum points:

- |                      |   |                         |
|----------------------|---|-------------------------|
| 1 $y = x^2 + 2x - 2$ | ⇒ | $(-1, -3)$ or $(-1, 3)$ |
| 2 $y = x^2 + 4x - 1$ |   | $(-2, 5)$ or $(-2, -5)$ |
| 3 $y = x^2 - 4x + 5$ |   | $(2, 1)$ or $(2, -1)$   |

6 Two graphs are shown:

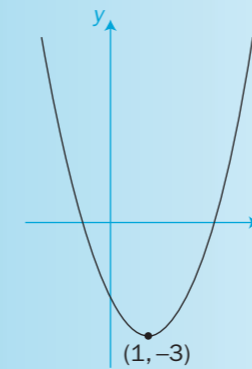


Fig. 1

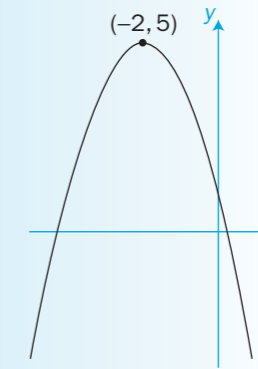


Fig. 2

Find the equation of each graph giving your answer in the form  $y = ax^2 + bx + c$ .

For Fig. 2, what would the equation be if the point  $(-2, 5)$  was a minimum instead of a maximum?

### 3.3 The quadratic formula

You can apply the method of completing the square to the general quadratic equation  $ax^2 + bx + c = 0$

$$ax^2 + bx + c = 0 \quad a \neq 0$$

Rewrite as:  $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$

Hence  $\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$  Dividing both sides by  $a$ .

Rearrange:  $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Complete the square:  $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

Factorise:  $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Take square roots:  $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

Make  $x$  the subject:  $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- You can solve quadratic equations of the form  $ax^2 + bx + c = 0$  using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You need to learn this formula. You can quote it without giving the proof.

The quadratic formula is particularly useful when a quadratic equation doesn't factorise easily.

Use the formula to solve the equation  $x^2 - 2x - 5 = 0$

Compare  $x^2 - 2x - 5 = 0$  with  $ax^2 + bx + c = 0$ :  
 $a = 1, b = -2, c = -5$

Use the formula:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Put in the values:  $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)}$

Simplify:  $x = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$

The solutions are  $x = 1 + \sqrt{6}$  or  $x = 1 - \sqrt{6}$

You should first try to factorise as it is quicker.

Take care with the signs.  
 If  $b$  is negative then  $-b$  is positive.  
 $b^2$  is always positive.

$$\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$$

Remember to divide the whole of the numerator by 2, not just the square root.

Leave your answer in surd form unless the question indicates otherwise.

You can often use quadratic equations to solve word problems.

EXAMPLE 2

Three times a number subtracted from the reciprocal of that number is equal to 4.  
 Find the possible numbers in surd form.

First express the word problem as an equation using algebra.

Let  $x$  be the number:  $\frac{1}{x} - 3x = 4$

So  $3x^2 + 4x - 1 = 0$

Compare with  $ax^2 + bx + c = 0$ :  $a = 3, b = 4, c = -1$

Use the formula:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Put in the values:  $x = \frac{-4 \pm \sqrt{(4)^2 - 4(3)(-1)}}{2(3)}$

Evaluate:  $x = \frac{-4 \pm \sqrt{16 + 12}}{6}$

Simplify:  $x = \frac{-4 \pm \sqrt{28}}{6}$

$$x = \frac{-4 \pm 2\sqrt{7}}{6}$$

Cancel down:  $x = \frac{-2 \pm \sqrt{7}}{3}$

The solutions are  $x = \frac{-2 + \sqrt{7}}{3}$  or  $x = \frac{-2 - \sqrt{7}}{3}$

Multiply each term by  $x$  and rearrange.

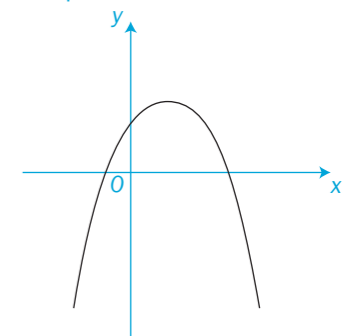
$$\sqrt{28} = \sqrt{4 \times 7} = \sqrt{4} \times \sqrt{7} = 2\sqrt{7}$$

#### The discriminant

- In the quadratic function  $f(x) = ax^2 + bx + c$  the expression  $(b^2 - 4ac)$  is known as the **discriminant**.

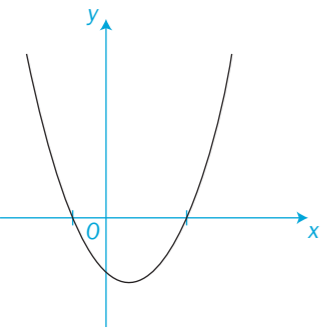
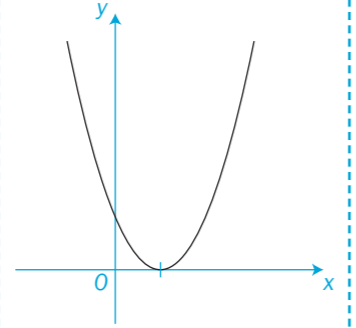
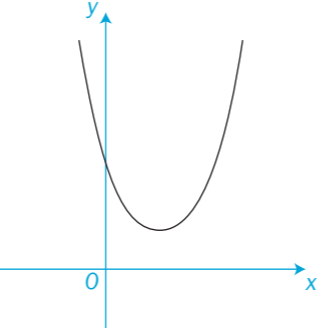
- You can use the discriminant to investigate the nature of the roots of a quadratic equation (two, one or no real roots).

If the coefficient of  $x^2$  is negative, then the parabola will be upside down.



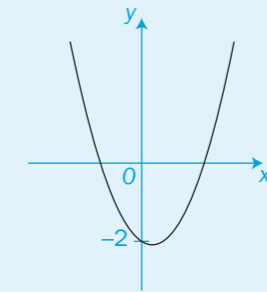
There are three possible cases for the solution of a quadratic equation.

The value of  $(b^2 - 4ac)$  tells you what type of roots the equation has.

<p><math>b^2 - 4ac &gt; 0</math></p>  <p>The curve cuts the <math>x</math>-axis in two places. There are two real roots.</p>	<p><math>b^2 - 4ac = 0</math></p>  <p>The <math>x</math>-axis is a <b>tangent</b> to the curve. There is one real root.</p>	<p><math>b^2 - 4ac &lt; 0</math></p>  <p>The curve never touches the <math>x</math>-axis. There are no real roots.</p>
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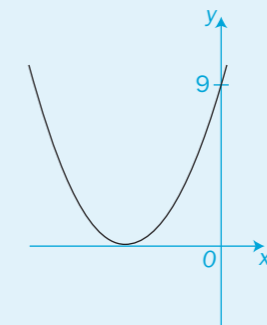
EXAMPLE 3 (CONT.)

**b** Write down the values of  $a, b$  and  $c$ :  
 $a = 1, b = -1, c = -2$   
 $b^2 - 4ac = 1 - -8$   
 $= 9$   
 $b^2 - 4ac > 0$   
 so two real roots



$f(0) = -2, f(1) = -2$

**c** Write down the values of  $a, b$  and  $c$ :  
 $a = 1, b = 6, c = 9$   
 $b^2 - 4ac = 36 - 36$   
 $= 0$   
 $b^2 - 4ac = 0$   
 so one real root



$f(0) = 9, f(-1) = 4$

Sometimes you can use knowledge of the roots to work out the equation of a quadratic function.

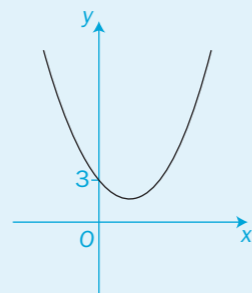
EXAMPLE 3

By finding the value of the discriminant, determine the nature of the roots of these equations.

- a**  $x^2 - 2x + 3 = 0$
- b**  $x^2 - x - 2 = 0$
- c**  $x^2 + 6x + 9 = 0$

Sketch a graph to illustrate each case.

**a** Write down the values of  $a, b$  and  $c$ :  
 $a = 1, b = -2, c = 3$   
 $b^2 - 4ac = 4 - 12$   
 $= -8$   
 $b^2 - 4ac < 0$   
 so no real roots



To sketch a graph, first find  $f(0)$  and another simple value  
 e.g.  $f(1)$ :  
 $f(0) = 3, f(1) = 2$

Example 3 continues on the next page.

EXAMPLE 4

- a** The equation  $2x^2 + kx + 3k = 0$  has one (repeated) root. Find the value of  $k$ .
- b** The equation  $x^2 + 5x + k = 0$  has two real and distinct roots. Find the range of possible values of  $k$ .

**a** For one (repeated) root  $b^2 - 4ac = 0$   
 $2x^2 + kx + 3k = 0$   
 $k^2 - 4 \times 2 \times 3k = 0$   
 Simplify:  $k^2 - 24k = 0$   
 Factorise:  $k(k - 24) = 0$   
 Solve:  $k = 0$  or  $k = 24$

$a = 2, b = k, c = 3k$

**b** For two roots  $b^2 - 4ac > 0$   
 $x^2 + 5x + k = 0$   
 $5^2 - 4 \times 1 \times k > 0$   
 Simplify:  $25 - 4k > 0$   
 $-4k > -25$   
 Divide both sides by  $-4$ :  $k < \frac{25}{4}$

$a = 1, b = 5, c = k$

Remember to change the direction of the inequality sign when dividing by a negative number.

15

**Exercise 3.3**

1 Solve these equations by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- |                       |                       |                        |
|-----------------------|-----------------------|------------------------|
| a $x^2 + 7x + 10 = 0$ | b $x^2 + 2x - 3 = 0$  | c $x^2 - 7x + 12 = 0$  |
| d $x^2 - 2x - 8 = 0$  | e $x^2 + 4x + 3 = 0$  | f $2x^2 - 3x - 2 = 0$  |
| g $2x^2 + x - 3 = 0$  | h $6x^2 + 5x + 1 = 0$ | i $8x^2 - 10x + 3 = 0$ |

2 Rearrange each equation, where necessary, into the form  $ax^2 + bx + c = 0$  and use the quadratic formula to obtain solutions where possible. Give your solutions in simplified surd form.

- |                       |                       |                       |
|-----------------------|-----------------------|-----------------------|
| a $x^2 + 3x + 1 = 0$  | b $x^2 - 3x - 2 = 0$  | c $x^2 - 5x + 3 = 0$  |
| d $2x^2 + 6x + 1 = 0$ | e $2x^2 + 3x - 4 = 0$ | f $3x^2 - x + 1 = 0$  |
| g $x^2 + 10x + 4 = 0$ | h $2 - 4x - x^2 = 0$  | i $(x + 1)^2 = x + 2$ |

3 Find the value of the discriminant,  $(b^2 - 4ac)$ , and decide whether each equation has one, two or no roots.

- |                       |                        |
|-----------------------|------------------------|
| a $x^2 + 4x + 5 = 0$  | b $x^2 - 3x - 10 = 0$  |
| c $x^2 + 6x + 9 = 0$  | d $x^2 - 2x - 1 = 0$   |
| e $2x^2 + 3x - 5 = 0$ | f $3x^2 - x - 1 = 0$   |
| g $4x^2 - 5x + 2 = 0$ | h $-2x^2 + 3x - 1 = 0$ |

4 Find the range of possible values of  $k$  if the following equations have

- i no roots
- ii one (repeated) root
- iii two real and distinct roots

- |                       |
|-----------------------|
| a $2x^2 - x + k = 0$  |
| b $kx^2 - 4x - 1 = 0$ |
| c $x^2 + k = 0$       |

Look at the cases where a solution is not possible – try to think why.

distinct means not equal

5 The difference between two numbers  $a$  and  $b$  is 1, and the product of the two numbers is 9. Show that there are two possible pairs of values which satisfy these conditions and evaluate the numbers.

6 The sum of a number and its reciprocal is 4. What are the two possible numbers which satisfy this condition?

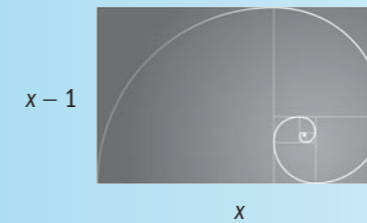
7 In a right-angled triangle the shorter sides are of lengths  $(x + 3)$  and  $(2x - 1)$  units and the length of the hypotenuse is  $3x$  units. Find the lengths of the sides of the triangle.

8 The equation  $ax^2 + bx + c = 0$  is such that  $-b = p$ ,  $2a = r$  and the discriminant is equal to  $q$  where  $p$ ,  $q$  and  $r$  are constants. Show that  $c = \frac{p^2 - q}{2r}$ .

9 Given that the quadratic equation  $ax^2 + bx + c = 0$  has two solutions, show that the quadratic equation given by  $a^2x^2 = b^2x - 2acx - c^2$ , where  $a$ ,  $b$  and  $c$  are constants and  $b \neq 0$ , also has two solutions.

**INVESTIGATIONS**

10 The golden ratio is a mathematically interesting number that supposedly represents divine proportion. The golden spiral, occurring in nature, is formed from the golden ratio.



The rectangle framing the golden spiral has sides in the golden ratio if the area of the rectangle is equal to one.

- a Write an equation using this information.
- b Hence solve to find  $x$ , giving your answer in surd form.

11 Choose values of  $k$  to make each of these quadratic equations have just one real (repeated) root.

- |                      |                      |                       |
|----------------------|----------------------|-----------------------|
| a $x^2 + kx + 4 = 0$ | b $x^2 + kx + 9 = 0$ | c $x^2 + kx + 16 = 0$ |
|----------------------|----------------------|-----------------------|

Can you generalise to a quadratic of the form  $x^2 + kx + n^2 = 0$ ?

12 Which integer values for  $k$  result in the following equation having two real roots?

$$x^2 + kx + 5 = 0$$

## Review 3

1 Solve these quadratic equations giving all values of  $x$ .

- a  $(x - 3)(x + 7) = 0$   
 b  $x(x + 4) = 0$   
 c  $5(x - 3)^2 = 0$   
 d  $(3x - 1)(2x - 3) = 0$

2 Factorise these equations and then solve to find the value(s) of  $x$ .

- a  $x(2x - 1) - 3(2x - 1) = 0$   
 b  $4x^2 - 25 = 0$   
 c  $x^2 + 6x - 7 = 0$   
 d  $2x^2 + 5x + 3 = 0$   
 e  $\frac{4}{9}x^2 - 1 = 0$   
 f  $10x^2 = 12 - 7x$   
 g  $5x - 4x^2 = 0$   
 h  $12x^2 + 12x - 9 = 0$

3 Use the method of completing the square to solve these equations giving your answers in simplified surd form.

- a  $x^2 + 4x + 2 = 0$   
 b  $2x^2 + 6x - 1 = 0$   
 c  $4 - 2x - x^2 = 0$   
 d  $(2x - 3)^2 = x + 1$

4 Use the formula to solve these quadratic equations leaving your answers in simplified surd form where appropriate.

- a  $x^2 - 3x - 3 = 0$   
 b  $2x^2 = 4x + 1$   
 c  $(x + 2)^2 - (2x + 3)^2 = 0$   
 d  $\frac{3}{4}(x^2 - 2x + 3) = x + 1$

5 Factorise completely  $x^3 - 7x^2 + 12x$

6 The equation  $ax^2 + 8x + a = 0$ , where  $a$  is a positive constant, has equal roots.  
Find the value of  $a$ .

7 Given that  $x^2 - 4x - 9 = (x + a)^2 + b$ , where  $a$  and  $b$  are constants

- a find the value of  $a$  and the value of  $b$   
 b Show that the roots of  $x^2 - 4x - 9 = 0$  can be written in the form  $p \pm q\sqrt{13}$  and hence determine the value of the integers  $p$  and  $q$ .

8 Given that  $x^2 + 10x + 36 \equiv (x + a)^2 + b$ , where  $a$  and  $b$  are constants:

- a find the value of  $a$  and the value of  $b$   
 b hence show that the equation  $x^2 + 10x + 36$  has no real roots.

The equation  $x^2 + 10x + k = 0$  has equal roots.

- c Find the value of  $k$ .  
 d For this value of  $k$ , sketch the graph of  $y = x^2 + 10x + k$ , showing the coordinates of any points at which the graph meets the coordinate axes.

[(c) Edexcel Limited 2003]

9 a The equation  $x^2 + ax + 12 = 0$  has equal roots.  
Find the values of  $a$ .

b The equation  $kx^2 - 4x + k = 0$  has equal roots.  
Find the possible values of  $k$ .

c The equation  $x^2 + 2kx + 2x + 7k - 3 = 0$  has repeated roots.  
Find the possible values of  $k$ .

10 a Solve the equation  $4x^2 + 12x = 0$

$f(x) = 4x^2 + 12x + c$ , where  $c$  is a constant.

b Given that  $f(x) = 0$  has equal roots, find the value of  $c$  and hence solve  $f(x) = 0$

[(c) Edexcel Limited 2003]



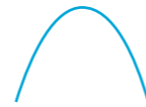
### Summary

Refer to

- A quadratic function of the form  $f(x) = ax^2 + bx + c$  has the shape



when  $a > 0$   
and



when  $a < 0$

minimum point

maximum point

3.1

- Quadratic equations of the form  $ax^2 + bx + c = 0$  can be solved by

- factorising
- completing the square or
- applying the formula

3.1

3.2

3.3

- The formula for solving the quadratic equation  $ax^2 + bx + c = 0$

$$\text{is } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.4

- The discriminant,  $b^2 - 4ac$ , may be used to determine how many roots a quadratic has.

- $b^2 - 4ac > 0$  (two distinct real roots)
- $b^2 - 4ac \geq 0$  (one or two real roots)
- $b^2 - 4ac = 0$  (one real root)
- $b^2 - 4ac < 0$  (no real roots)

3.4

### Links

Quadratic equations play a role in modelling the motion of objects ranging from the astronomical down to the subatomic scale.

Depending on how you slice a cone, you get four different **conic sections** – circle, ellipse, hyperbola and parabola.

These are all described by equations of degree 2 and they all have practical applications. You will be most familiar with circles, but ellipses and hyperbolas describe the motion of planets around the sun, and parabolas are used in the design of radio telescopes and satellite dishes.



On a more down-to-earth level, each time you throw a stone or kick a ball the stone or ball traces a parabola (with its own quadratic equation) in the air.