

7

Parametric equations

This chapter will show you how to

- sketch curves using their parametric equations
- convert parametric equations to Cartesian equations
- find points of intersection of curves and lines using parametric equations
- differentiate parametric equations to find equations of tangents and stationary values
- integrate parametric equations to find areas under curves.

Before you start

You should know how to:

1 Substitute into formulae.

e.g. If $a = 2x + 3$ and $b = 1 - 4x$,

find y when $y = a^2 - b$

Substitute for a and b :

$$\begin{aligned} y &= (2x + 3)^2 - (1 - 4x) \\ &= 4x^2 + 12x + 9 - 1 + 4x \\ &= 4x^2 + 16x + 8 \end{aligned}$$

2 Solve simultaneous equations.

e.g. Solve $y = x + 1$ and $y + 5 = x^2$

Eliminate y : $(x + 1) + 5 = x^2$

$$x^2 - x - 6 = 0$$

$$(x + 2)(x - 3) = 0$$

So, $x = -2$ or 3 and $y = -1$ or 4

The solutions are $(-2, -1)$ and $(3, 4)$.

3 Differentiate and integrate functions.

e.g. Find $\frac{dy}{dx}$ and $\int y dx$ if $y = \left(1 + \frac{1}{x^2}\right)^2$

Expand the brackets:

$$y = 1 + \frac{2}{x^2} + \frac{1}{x^4} = 1 + 2x^{-2} + x^{-4}$$

$$\text{Hence } \frac{dy}{dx} = 2(-2)x^{-3} + (-4)x^{-5}$$

$$= -\frac{4}{x^3} - \frac{4}{x^5}$$

$$\text{and } \int y dx = x + 2\frac{x^{-1}}{-1} + \frac{x^{-3}}{-3} + c$$

$$= x - \frac{2}{x} - \frac{1}{3x^3} + c$$

Check in:

1 If $m = 2(x + 1)$ and $n = \frac{3x - 2}{4}$
find y when

a $y = \frac{1}{4}m + 2n$

b $y = 2m^2 + 16n^2$

2 Solve these simultaneous equations.

a $2x + 3y = 1, y + 3x = 5$

b $x^2 + y^2 = 3, x + 2y = 1$

3 Find $\frac{dy}{dx}$ and $\int y dx$ when

a $y = x^2 + 1 + \frac{1}{x^2}$

b $y = x(1 + x)\left(1 + \frac{1}{x}\right)$

c $y = \frac{x^2 + 1}{x^2}$

7.1

Parametric equations and curve sketching

A **Cartesian equation** has the form $y = f(x)$

e.g. $y = x^2 + 2x + 1$

Some relationships between x and y involve a third variable.

This third variable is called a **parameter**.

The equations $x = f(t)$, $y = g(t)$ are called **parametric equations**.

t is the parameter.

You can sketch a curve described by parametric equations by finding points on the graph for a range of values of t .

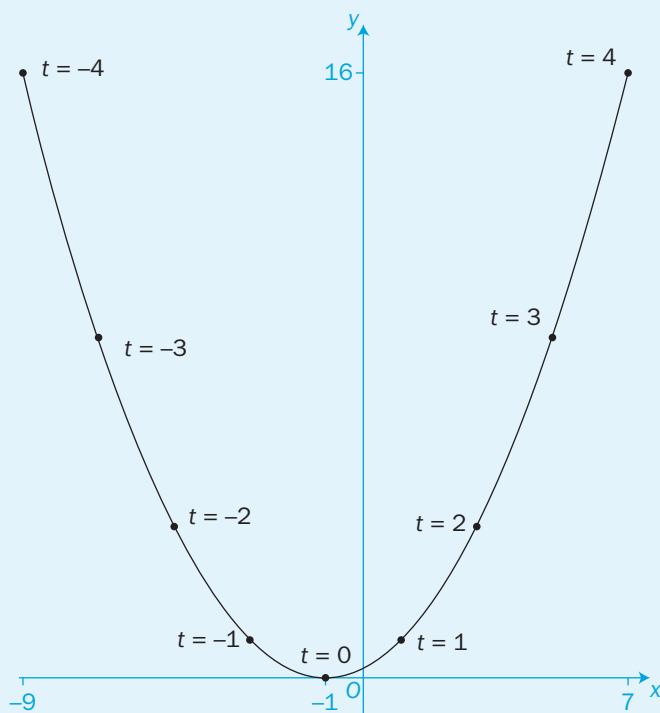
Each point on the graph has a value of t associated with it.

EXAMPLE 1

Sketch the graph of the curve with the parametric equations
 $x = 2t - 1$, $y = t^2$ for $-4 \leq t \leq 4$

Construct a table of values:

t	-4	-3	-2	-1	0	1	2	3	4
x	-9	-7	-5	-3	-1	1	3	5	7
y	16	9	4	1	0	1	4	9	16



The curve is a parabola.

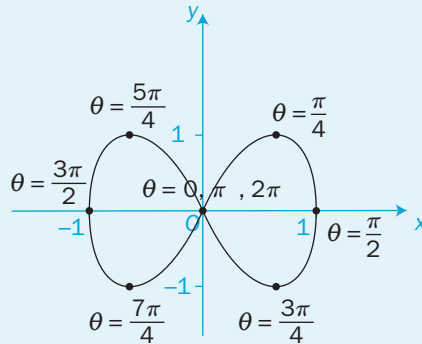
It is useful to label each point with its value of t . You can then see how the curve takes shape as t varies.

EXAMPLE 2

Sketch the curve given parametrically by the equations
 $x = \sin \theta, y = \sin 2\theta$ for $0 \leq \theta \leq 2\pi$

Construct a table of values for $0 \leq \theta \leq 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	0	0.707	1	0.707	0	-0.707	-1	-0.707	0
y	0	1	0	-1	0	1	0	-1	0



θ is the parameter.

θ is in radians.

As θ increases from 0, the curve traces the two loops of a figure-of-eight.

When a curve is expressed using parametric equations, you can find the Cartesian equation by eliminating the parameter t (or θ).

EXAMPLE 3

Find the Cartesian equation of the curves which have these parametric equations.

a $x = 2t - 1, y = 8t^2 + 3$ **b** $x = 2\sin \theta + 3, y = 2\cos \theta - 5$

a Substitute t from $x = 2t - 1$ into the equation for y :

$$\begin{aligned} \text{From } x = 2t - 1, t = \frac{x+1}{2}, \text{ so } y &= 8\left(\frac{x+1}{2}\right)^2 - 1 \\ &= 2(x+1)^2 - 1 \\ &= 2x^2 + 4x + 1 \end{aligned}$$

The Cartesian equation is $y = 2x^2 + 4x + 1$

b Find $\sin \theta$ and $\cos \theta$ in terms of x and y :

$$\text{From } x = 2\sin \theta + 3, \quad \sin \theta = \frac{x-3}{2}$$

$$\text{From } y = 2\cos \theta - 5, \quad \cos \theta = \frac{y+5}{2}$$

Substitute into $\sin^2 \theta + \cos^2 \theta = 1$:

$$\begin{aligned} \left(\frac{x-3}{2}\right)^2 + \left(\frac{y+5}{2}\right)^2 &= 1 \\ (x-3)^2 + (y+5)^2 &= 4 \end{aligned}$$

The Cartesian equation is $(x-3)^2 + (y+5)^2 = 4$

$2x^2 + 4x + 1$ is a quadratic expression, which indicates that the curve is a parabola.

This equation represents a circle, centre $(3, -5)$ and radius $\sqrt{4} = 2$. See **C2** for revision.

You can also find parametric equations of a curve represented by a Cartesian equation.

EXAMPLE 4

Find parametric equations for the curve with the Cartesian equation $y = 6x\sqrt{1-x^2}$

You need to find a parameter which will simplify $\sqrt{1-x^2}$

Recall that $1 - \sin^2 \theta = \cos^2 \theta$

Let $x = \sin \theta$:

$$\begin{aligned} \text{So } y &= 6 \sin \theta \sqrt{1 - \sin^2 \theta} = 6 \sin \theta \cos \theta \\ &= 3(2 \sin \theta \cos \theta) \\ &= 3 \sin 2\theta \end{aligned}$$

Hence, parametric equations for the curve are
 $x = \sin \theta, y = 3 \sin 2\theta$

The parameter is θ .

Letting $x = \cos \theta$ will also give
 $y = 3 \sin 2\theta$. Try this yourself.

There may be more than one possible pair of parametric equations for a given curve.

Exercise 7.1

- 1 A curve has the parametric equations $x = 3t, y = t^2 - 3$

Copy and complete this table.

Hence sketch the graph of the curve for $-3 \leq t \leq 3$

t	-3	-2	-1	0	1	2	3
x							
y							

- 2 The parametric equations of a curve are $x = 3t^2, y = t^3$

Copy and complete this table.

Hence sketch the graph of the curve for $-2 \leq t \leq 2$

t	-2	-1	0	1	2
x					
y					

- 3 Construct your own tables of values to sketch the graphs of the curves with these parametric equations for the range of values given in each case.

a $x = t^2 - 4, y = \frac{1}{2}t^3$ for $-3 \leq t \leq 3$

b $x = t^3 - 2t + 4, y = t - 1$ for $-2 \leq t \leq 2$

c $x = t^2, y = \frac{1}{t}$ for $-3 \leq t \leq 3$

d $x = 4 \sin \theta, y = 4 \cos \theta$ for $0 \leq \theta \leq 2\pi$

e $x = 5 \cos \theta, y = 3 \sin \theta$ for $0 \leq \theta \leq 2\pi$

f $x = \sec \theta, y = \tan \theta$ for $0 \leq \theta \leq 2\pi$

- 4 Find the Cartesian equation for each of the curves given parametrically by these equations.

a $x = t + 4$ $y = 1 - 2t$

b $x = \frac{3}{t}$ $y = 4t$

c $x = t + 1$ $y = t^2 - 2$

d $x = t^2$ $y = t^3$

e $x = t^2 - 1$ $y = t^3 + 2$

f $x = t^2$ $y = \frac{2}{t}$

g $x = \frac{1-t}{t}$ $y = \frac{1+t}{t}$

h $x = 3\cos \theta$ $y = 4\sin \theta$

i $x = \sin \theta$ $y = \cos 2\theta$

j $x = 3\cos \theta$ $y = 5\cos 2\theta$

k $x = 2\sec \theta$ $y = 3\tan \theta$

l $x = \frac{1+t}{1-t}$ $y = \frac{2+t}{1-t}$

- 5 Point P lies on the curve $x = 2t - 4$, $y = t + 1$.
If the y -coordinate of P is 6, find its x -coordinate.
- 6 Point Q lies on the curve $x = \frac{2+t}{2-t}$, $y = \frac{3-2t}{2-t}$.
If the x -coordinate of Q is 4, find its y -coordinate.
- 7 The point $(4, k)$ lies on the curve $x = t^2 - 5$, $y = t - 1$.
Find the possible values of k .
- 8 The variable point $P(at, t^2 - 1)$ meets the line $y = 8$ at the point $(6, 8)$.
Find the possible values of a and the Cartesian equation of the curve along which P moves.
- 9 Find the coordinates of the points where these curves meet the x -axis.
- a** $x = t^2 + 1$ $y = t - 3$ **b** $x = 1 + t^3$ $y = 2 - t$
- c** $x = 5t + 3$ $y = t^2 - 4$ **d** $x = 3\cos \theta$ $y = \sin \theta$
- 10 Find the coordinates of the points where these curves meet the y -axis.
- a** $x = t - 5$ $y = t^2 - 2$ **b** $x = t^2 - 3t + 2$ $y = t + 4$
- c** $x = t^3 - t$ $y = t^2$ **d** $x = \tan \theta$ $y = \sec \theta$
- 11 The curve $x = at^2 - 3$, $y = a(t - 2)$ contains the point $(17, 0)$.
Find the value of a .
- 12 The point $(20, 40)$ lies on the curve $x = at^2$, $y = 4at$.
Find the value of a .

- 13** The curve $x = 2a \sin \theta$, $y = 1 + a \cos 2\theta$ intersects the y -axis at the point $(0, 4)$.

Find the value of a .

- 14** Points A and B lie on the curve $x = t^2 - 3$, $y = 2t + 3$ where $t = 2$ and $t = 3$ respectively.

Find

- the distance between A and B
- the gradient of the chord AB
- the equation of the chord AB .

- 15** Show that these two pairs of parametric equations represent the same straight line. Find the Cartesian equation of the line.

a $x = 1 - t$ $y = 3 - 2t$

b $x = \frac{1}{t+1}$ $y = \frac{t+3}{t+1}$

- 16** Find parametric equations of the curve with the Cartesian equation

$$y = x\sqrt{4 - x^2}$$

if θ is the parameter such that $x = 2\cos \theta$

- 17** Use the identity $1 + \tan^2 \theta = \sec^2 \theta$ to find parametric equations for the curve with the Cartesian equation $y = \frac{x}{\sqrt{1+x^2}}$

- 18** The Cartesian equation of a curve is $y = \frac{3\sqrt{1-x^2}}{x}$

Find parametric equations for this curve if

a $x = \sin \theta$ **b** $x = \frac{1}{t}$

- 19** The equation of a circle is $x^2 + y^2 - 6x - 4y + 12 = 0$

Refer to Example 3.

- a** The equation is written in the form $(x - \alpha)^2 + (y - \beta)^2 = 1$
Find the values of α and β .

- b** Hence, find parametric equations for the circle in terms of the parameter θ .

- 20** A hyperbola has the equation $9x^2 - 4y^2 - 18x + 16y - 43 = 0$

- a** The equation is written in the form $\frac{(x-a)^2}{b^2} - \frac{(y-c)^2}{d^2} = 1$

Find the values of a , b , c and d .

- b** Hence, find parametric equations for the hyperbola in terms of θ .

- 21** A curve has parametric equations
 $x = t - 2\sin t$, $y = 1 - 2\cos t$, $0 \leq t \leq 2\pi$
- Find the values of t , in terms of π , at the two points where the curve crosses the x -axis.
 - The curve crosses the y -axis at two points where $t = \alpha$ and $t = \beta$. Show that one of these points has $\alpha = 0$. Find, by trial-and-improvement, the value of β to 1 decimal place. Find the coordinates of these two points on the y -axis.
- 22** By substituting $y = tx$, find parametric equations for the curves with these Cartesian equations.
- $y^3 = x^2$
 - $y = x^2 - 2x$
 - $x^3 - y^3 = x^2$
 - $x - y = xy$

- 23** A curve has the Cartesian equation $x^3 + y^3 = 3xy$
- Show, by substituting $y = tx$, that the curve can be represented by the parametric equations

$$x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$$
 - Find the points where $t = 0$ and $t = \infty$
 - Investigate the curve when t is close to -1 .
 - Hence, sketch the curve and find the equation of any asymptote.

INVESTIGATION

- 24 a** Use a computer's graphical package to check your answers to questions **1**, **2** and **3**. You can also check answers to other questions by drawing appropriate graphs.
- b** Investigate how changing the values of constants A , B , m and n in these parametric equations alters the graphs of the curves.

$$x = A \sin m\theta, \quad y = B \cos n\theta \quad \text{for } 0 \leq \theta \leq 2\pi$$

Your computer software may need to have θ in degrees, that is, $0^\circ \leq \theta \leq 360^\circ$

You can use simultaneous equations to find the points of intersection when a curve is expressed in parametric equations.

EXAMPLE 1

Find the points of intersection of the curve with parametric equations $x = 2t - 1$, $y = t^2$ and the straight line $y = 3x - 2$

Solve the equations $x = 2t - 1$, $y = t^2$ } simultaneously.
 $y = 3x - 2$ }

Substitute $x = 2t - 1$, $y = t^2$ into $y = 3x - 2$ to eliminate x and y :

$$\begin{aligned} t^2 &= 3(2t - 1) - 2 \\ t^2 - 6t + 5 &= 0 \\ (t - 1)(t - 5) &= 0 \quad \text{so } t = 1 \text{ or } 5 \end{aligned}$$

When $t = 1$, $x = 2(1) - 1 = 1$ and $y = 1^2 = 1$

When $t = 5$, $x = 2(5) - 1 = 9$ and $y = 5^2 = 25$

The points of intersection are $(1, 1)$ and $(9, 25)$.

There are three unknowns, x , y and t .

EXAMPLE 2

Find the points A , B and C where the curve given parametrically by $x = t^2 - 4$, $y = t - 1$ intersects the two coordinate axes. Hence, find the area of triangle ABC .

When $y = 0$, $t - 1 = 0$ giving $t = 1$

When $t = 1$, $x = 1^2 - 4 = -3$,
giving the point of intersection $A(-3, 0)$.

When $x = 0$, $t^2 - 4 = 0$ giving $t = \pm 2$

When $t = 2$, $y = 2 - 1 = 1$

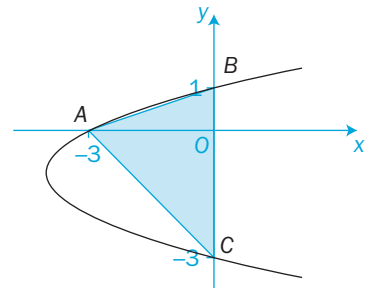
and, when $t = -2$, $y = -2 - 1 = -3$,
giving the points of intersection $B(0, 1)$ and $C(0, -3)$.

The area of triangle ABC is

$$\frac{1}{2}BC \times OA = \frac{1}{2} \times 4 \times 3 = 6 \text{ square units}$$

The curve meets the x -axis when $y = 0$

The curve meets the y -axis when $x = 0$



Exercise 7.2

- Find the points of intersection of the parabola $x = t^2$, $y = 2t$ and the straight lines
 - $x + y = 3$
 - $4x + 2y = 15$

- 2 Find the points of intersection of each curve and the given line.
- a** $x = t^2 - 1, y = 2t + 1$ $y = x + 2$ **b** $x = t^3, y = t^2 + 2t$ $y = 2x + 1$
- c** $x = t^2 - 1, y = t^2 + t + 1$ $2y - x - 3 = 0$
- 3 Find the points of intersection of the curve with parametric equations $x = 2t^2, y = 3t$ and the circle $x^2 + y^2 - 6x - 1 = 0$
- 4 Find the points of intersection of the parabola $x + y^2 = 9$ and the curve $x = (t - 3)^2, y = 2t$
- 5 Find the points where these curves cross the coordinate axes.
- a** $x = t - 1$ $y = t - 4$ **b** $x = t - 2$ $y = t^2 - 9$ **c** $x = t^2 + 1$ $y = t - 3$
- d** $x = t^3 - 1$ $y = t^2 - 4$ **e** $x = 1 - \frac{1}{t}$ $y = \frac{t+1}{t^2-4}$ **f** $x = \pi - 2t$ $y = 1 - \sin t$
- 6 The variable point $P(t^2, 2t)$ moves along a locus. Find the points where the locus crosses the straight line $y = 2x - 4$
- 7 The point $P(t^2, 4t)$ moves as t varies. Q is the midpoint of OP where O is the origin. Write down the coordinates of Q . Find the Cartesian equation of the locus of Q .
- 8 The point $P(2t^2, 6t)$ lies on a curve. The foot of the perpendicular from P to the x -axis is Q . The midpoint of PQ is M . Find
- a** the coordinates of Q and M in terms of t
- b** the Cartesian equation of the locus of M as P moves.
- 9 Find the points of intersection of the curve $x = 1 - 5t, y = t^3 + t^2$ and the line $x + y + 2 = 0$
- 10 The curve $x = t + 1, y = t^2 - k$ intersects the x - and y -axes at points P and Q respectively. Find the value of k ($k \neq 1$) such that $OP = 2OQ$ where O is the origin.

INVESTIGATION

- 11 Use a computer's graphical software to draw graphs using their parametric equations. Check your answers to the problems in this exercise where you have found points of intersection.

You can differentiate parametric equations to obtain $\frac{dy}{dx}$.

If $x = f(t)$ and $y = g(t)$

the **chain rule** gives $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Once you know $\frac{dy}{dx}$, you can use it to find equations of tangents and normals to a curve and to find stationary values.

EXAMPLE 1

A curve has parametric equations $x = t^3 + 2t + 4$, $y = t^2 - 1$

Find **a** the equation of the tangent at the point where $t = 2$

b the nature of any stationary values and the points at which they occur.

a Differentiate x wrt t : $\frac{dx}{dt} = 3t^2 + 2$

Differentiate y wrt t : $\frac{dy}{dt} = 2t$

So $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 + 2}$

Substitute $t = 2$: $= \frac{2 \times 2}{3 \times 2^2 + 2} = \frac{2}{7}$ $\frac{2}{7}$ is the gradient of the tangent when $t = 2$

When $t = 2$ $x = 2^3 + 2 \times 2 + 4 = 16$
and $y = 2^2 - 1 = 3$

So, the tangent passes through the point $(16, 3)$ with a gradient of $\frac{2}{7}$.

The equation of the tangent is $\frac{y - 3}{x - 16} = \frac{2}{7}$ $\frac{y - y_1}{x - x_1} = m$

Rearrange: $7y = 2x - 11$

See **C1** for revision.

b For stationary values, $\frac{dy}{dx} = \frac{2t}{3t^2 + 2} = 0$

So the only stationary value occurs when $t = 0$

at the point where $x = 0^3 + 2 \times 0 + 4 = 4$

and $y = 0^2 - 1 = -1$

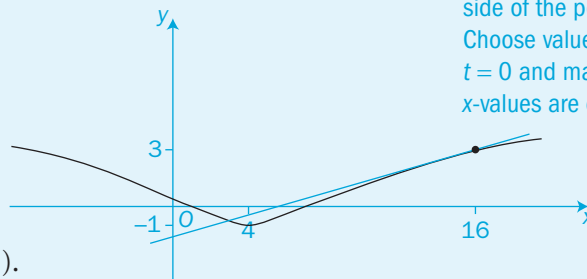
The numerator $2t = 0$ when $t = 0$

Investigate the gradient on either side of the point $(4, -1)$:

Choose values of t either side of $t = 0$ and make sure that the x -values are either side of $x = 4$

t	-1	0	1
x	1	4	7
$\frac{dy}{dx}$	$-\frac{2}{5}$	0	$\frac{2}{5}$

There is a minimum value at the point $(4, -1)$.



EXAMPLE 2

A curve is defined parametrically by $x = 3 - t$, $y = 4 - t - t^2$
 A normal is drawn to the curve at the point A where $t = 1$
 Find another point B at which this normal intersects the curve.

Let $t = 1$: $x = 3 - 1 = 2$ and $y = 4 - 1 - 1^2 = 2$
 So, the normal is drawn at the point $A(2, 2)$.

Find $\frac{dy}{dx}$ and substitute $t = 1$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-1 - 2t}{-1} \\ &= 1 + 2t\end{aligned}$$

When $t = 1$,
 the gradient of the tangent at the point $A(2, 2)$ is $1 + 2(1) = 3$

The gradient of the normal at the same point is $-\frac{1}{3}$.

So, the equation of the normal at the point $A(2, 2)$

is
$$\begin{aligned}\frac{y - 2}{x - 2} &= -\frac{1}{3} \\ 3y + x &= 8\end{aligned}$$

To find the intersections of the normal and curve, substitute the parametric equations into $3y + x = 8$:

$$\begin{aligned}3(4 - t - t^2) + (3 - t) &= 8 \\ 3t^2 + 4t - 7 &= 0 \\ (t - 1)(3t + 7) &= 0 \\ t &= 1 \text{ or } -\frac{7}{3}\end{aligned}$$

$t = 1$ gives the initial point A on the normal,

Hence at B , $t = -\frac{7}{3}$

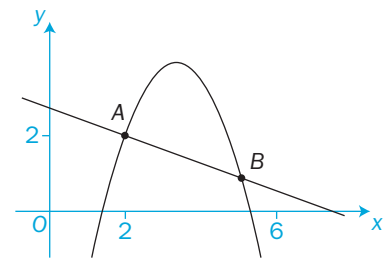
Substitute $t = -\frac{7}{3}$ into the parametric equations:

$$x = 3 + \frac{7}{3} = 5\frac{1}{3} \quad y = 4 + \frac{7}{3} - \frac{49}{9} = \frac{8}{9}$$

The required point of intersection is $(5\frac{1}{3}, \frac{8}{9})$.

If the gradients of tangent and normal are m and m' , then $m' = -\frac{1}{m}$

You know that the curve and normal intersect when $t = 1$



Exercise 7.3

1 Find the gradient of each curve with these parametric equations at the point with the given value of t (or θ).

a $x = t + 1$ $y = 3t^2$ when $t = 2$

b $x = t^2 - 2$ $y = t^3 + 1$ when $t = 4$

c $x = t^3 - t^2$ $y = (t + 3)^2$ when $t = 1$

d $x = 3\sin \theta$ $y = 5\cos \theta$ when $\theta = \frac{\pi}{4}$

e $x = 1 + t^2$ $y = 1 + \frac{1}{t}$ when $t = -2$

f $x = \sin 2\theta$ $y = \theta \cos \theta$ when $\theta = 0$

2 Find the equation of the tangent and the normal to the curves with these equations at the point where t (or θ) has the given value.

a $x = 2t^2$ $y = 4t$ when $t = \frac{1}{2}$

b $x = t^2 + 1$ $y = t^3 - 1$ when $t = 1$

c $x = 2\cos \theta$ $y = \cos 2\theta$ when $\theta = \frac{\pi}{4}$

d $x = \frac{2t}{1+t^2}$ $y = \frac{1-t^2}{1+t^2}$ when $t = 2$

3 Find the stationary points on these curves.

a $x = t$ $y = t^3 - t$

b $x = t^2$ $y = t + \frac{1}{t}$

c $x = \theta - \cos \theta$ $y = \sin \theta$ for $0 < \theta < 2\pi$

d $x = 3\sin \theta + 2$ $y = 3\cos \theta + 5$ for $0 \leq \theta \leq 2\pi$

4 The curve $x = 2t^2$, $y = 4t$ has a normal at point $P(8, 8)$.

Find the equation of the normal.

Also find the point where the normal meets the curve a second time.

5 Find the equation of the normal to the curve $x = 6t$, $y = \frac{6}{t}$ at the point where $t = 2$

Also find the point where the normal intersects the curve again.

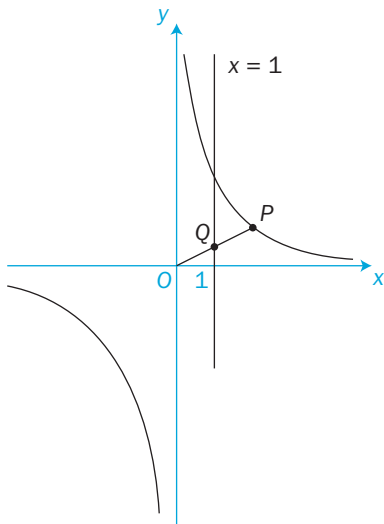
- 6 The tangent at point $P(1, 1)$ to the curve $x = \frac{1}{t}$, $y = t^2$ intersects the curve at point Q .

Find the equation of the tangent at P and the coordinates of Q .

- 7 The point P lies on the curve $x = 5\cos \theta$, $y = 4\sin \theta$
 A and B are the points $(-3, 0)$ and $(3, 0)$ respectively.

- a Find the distances AP and BP in terms of θ .
 b Show that the sum of the distances AP and BP is constant for all points P .

- 8 The line from the variable point $P\left(t, \frac{1}{t}\right)$ to the origin O intersects the line $x = 1$ at the point Q .



Find

- a the gradient and the equation of the line OP
 b the coordinates of Q in terms of t
 c the Cartesian equation of the locus of the midpoint of PQ .

INVESTIGATION

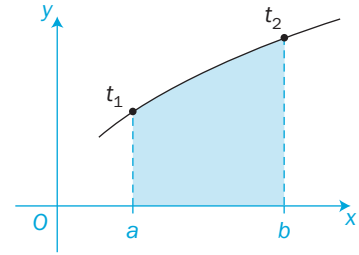
- 9 Investigate how to draw tangents and normals to curves using a computer's graphical package. Hence, check some of your answers to the problems in this exercise.

7.4 Integration

You can modify the expression $\int_a^b f(x) dx$ for the area under a curve using the chain rule.

For parametric equations $x = f(t)$, $y = g(t)$, the area under the curve between the points where $t = t_1$ and $t = t_2$ is given by

$$\int_{t_1}^{t_2} y \frac{dx}{dt} dt$$



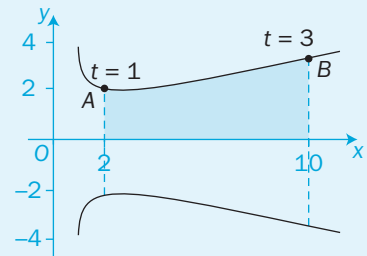
The limits of this integral are $t = t_1$ and $t = t_2$, as the independent variable is now t and not x .

EXAMPLE 1

This sketch shows the curve with parametric equation

$$x = t^2 + 1, \quad y = t + \frac{1}{t}$$

Find the shaded area between the curve and the x -axis from $x = 2$ to $x = 10$



Let $x = 2$: $t^2 = 1$, $t = \pm 1$ so $y = 2$ or -2 .

So, point A is $(2, 2)$.

Let $x = 10$: $t^2 = 9$, $t = \pm 3$ so $y = 3\frac{1}{3}$ or $-3\frac{1}{3}$.

So, point B is $(10, 3\frac{1}{3})$.

You want the area under the curve from $A(2, 2)$ where $t = 1$ to $B(10, 3\frac{1}{3})$ where $t = 3$

Integrate:

$$\text{Required area} = \int_2^{10} y dx = \int_1^3 y \frac{dx}{dt} dt \text{ where } y = t + \frac{1}{t} \text{ and } \frac{dx}{dt} = 2t$$

$$= \int_1^3 \left(t + \frac{1}{t}\right)(2t) dt$$

$$= 2 \int_1^3 (t^2 + 1) dt$$

$$= 2 \left[\frac{t^3}{3} + t \right]_1^3$$

$$= 2 \left(9 + 3 - \frac{1}{3} - 1 \right) = 21\frac{1}{3} \text{ square units}$$

You are calculating the area above the x -axis only.

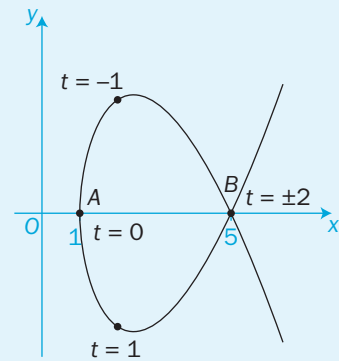
Notice the change in the limits as the independent variable changes from x to t .

EXAMPLE 2

This diagram shows the curve with parametric equations $x = t^2 + 1$, $y = t^3 - 4t$

Find the values of t at the points $A(1,0)$ and $B(5,0)$.

Find the area of the region enclosed by the loop of the curve.



Find t at A and B :

At points A and B , $y = 0$

$$t^3 - 4t = 0$$

$$t(t-2)(t+2) = 0$$

$$t = 0 \text{ or } \pm 2$$

Let $t = 0$: $x = 0^2 + 1 = 1$ and $y = 0 - 0 = 0$

So, $t = 0$ at the point $A(1,0)$.

Let $t = 2$: $x = 2^2 + 1 = 5$ and $y = 2^3 - 4 \times 2 = 0$

Let $t = -2$: $x = (-2)^2 + 1 = 5$ and $y = (-2)^3 - 4 \times (-2) = 0$

So, $t = \pm 2$ at the point $B(5,0)$.

You want the area under the curve from $A(1,0)$ where $t = 0$ to $B(5,0)$ where $t = \pm 2$

When $t = 1$, $x = 2$, $y = 1 - 4 = -3$, giving the point $(2, -3)$ below the x -axis.

So, integrating from $t = 0$ to $t = 2$ will give the area below the x -axis.

Similarly, $t = -1$ gives the point $(2, 3)$ and integrating from $t = 0$ to $t = -2$ gives the area above the x -axis.

area of loop = $2 \times$ area enclosed by the curve above the x -axis

$$= 2 \times \int_1^5 y dx = 2 \int_0^{-2} y \frac{dx}{dt} dt \text{ where } y = t^3 - 4t \text{ and } \frac{dx}{dt} = 2t$$

$$= 2 \int_0^{-2} (t^3 - 4t)(2t) dt$$

$$= 4 \int_0^{-2} (t^4 - 4t^2) dt$$

$$= 4 \left[\frac{t^5}{5} - \frac{4t^3}{3} \right]_0^{-2}$$

$$= 4 \left(-\frac{32}{5} + \frac{32}{3} - 0 \right)$$

$$= 17 \frac{1}{15} \text{ square units}$$

Find x and y when $t = 0$

Find x and y when $t = +2$
and $t = -2$.

The curve is symmetrical about the x -axis.

You could find the answer directly by integrating the whole way from $t = -2$ to $t = 2$ using

$$\int_{-2}^2 2t^4 - 8t^2 dt$$

Try this yourself.

Exercise 7.4

1 Find, in each case, the area between the x -axis and the curve between the two points P and Q defined by the given values of t or x .

a $x = t + 2$ $y = 3t - 1$ for $t = 1$ to $t = 4$

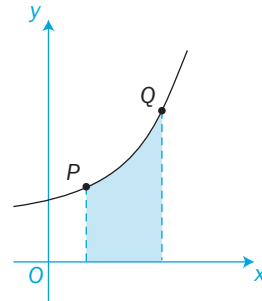
b $x = 2t$ $y = 4t^2 + 1$ for $t = 0$ to $t = 2$

c $x = t^2$ $y = 1 + 12t - 3t^2$ for $t = 0$ to $t = 4$

d $x = t - 1$ $y = 2t - 1$ for $x = 1$ to $x = 5$

e $x = 2t$ $y = t^2 + 2$ for $x = 2$ to $x = 4$

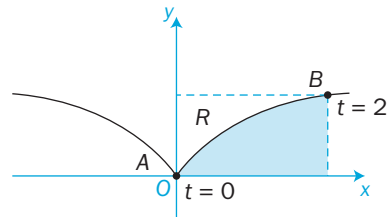
f $x = 4t$ $y = \frac{2}{t}$ for $x = 2$ to $x = 4$



2 A and B are the points on the curve $x = t^3$, $y = t^2$ where $t = 0$ and $t = 2$

Find the shaded area on the diagram.

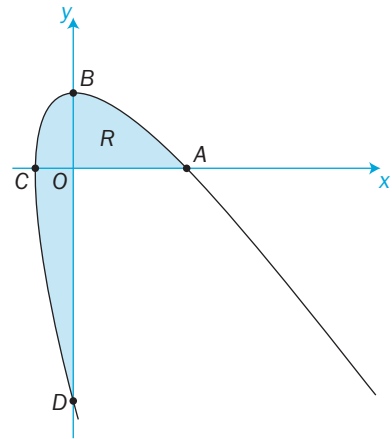
Also find the area of the region labelled R .



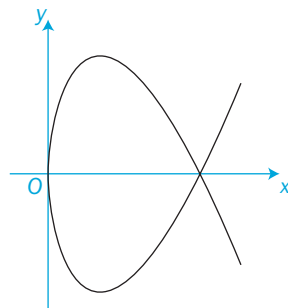
3 a The curve $x = t^2 - 1$, $y = 2t(2 - t)$ cuts the coordinate axes at the points A , B , C and D . Find the positions of these points and their associated t -values.

b Calculate the area of the region R in the first quadrant enclosed by the curve and the two coordinate axes.

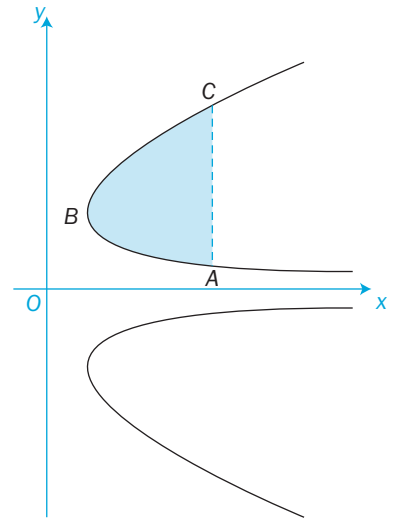
c Calculate the total shaded area.



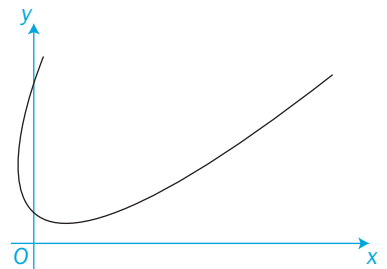
4 This diagram shows the curve $x = t^2$, $y = t(4 - t^2)$. Find the values of t at the two points where the curve cuts the x -axis. Hence, find the area enclosed by the loop.



- 5 a** This diagram shows the curve $x = t^2 + \frac{1}{t^2}$, $y = 2t$
- Find the coordinates of the points A , B and C on the curve where $t = \frac{1}{2}$, 1 and 2 respectively.
- b** Calculate the shaded area of the diagram bounded by the curve and the line AC .



- 6** The curve $x = t^2 - 3t + 2$, $y = 4t^2 + 1$ is shown on this diagram.
- Find the points at which the curve cuts the y -axis.
- Also find the area bounded by the curve and the line $y = 5$



- 7 a** A curve is expressed parametrically by $x = t^2 + 1$, $y = 2t$
- Another curve has the parametric equations $x = 2s$, $y = \frac{2}{s}$
- Find any points of intersection.
- b** Find the area enclosed by the two curves and the line $x = 5$
- 8 a** The straight line $y = c - 2x$ touches the curve $x = t$, $y = \frac{1}{t}$ at point P .
- Find the possible values of c and the coordinates of P .
- b** Find the area enclosed by the curve $x = t$, $y = \frac{1}{t}$ and the line $y = 3 - 2x$

INVESTIGATION

- 9** Show that, when finding the area under a curve, you get the same answer whether you use the curve's Cartesian equation or its parametric equations.
- Consider some of the curves in questions **1** and **2** of this exercise.

Review 7

- 1 Find the Cartesian equation for each of the curves given parametrically by these equations.

a $x = t - 1, \quad y = t^2 + 1$ **b** $x = \frac{t-1}{t}, \quad y = \frac{t+1}{t}$

c $x = 4\cos \theta, \quad y = 3\sin \theta$ **d** $x = 2\cos \theta, \quad y = \cos 2\theta$

- 2 Use the identity $\sin^2 \theta + \cos^2 \theta \equiv 1$ to find parametric equations for the curves with the Cartesian equation

a $y = 3x\sqrt{1-x^2}$ **b** $y = \frac{5x}{\sqrt{4-x^2}}$

- 3 The point $(5, a)$ lies on the curve $x = t^2 + 1, \quad y = \frac{1}{3}(t - 1)$
Find the possible values of a .

- 4 Find the points where the curve given by these parametric equations

a $x = 3t + 1, \quad y = t^2 - 1$ intersects the straight line $2x + y = 6$

b $x = t^3 - 4, \quad y = t^2 - 4$ intersects the x -axis

c $x = \cos t, \quad y = \sqrt{5}\sin t$ intersects the circle $x^2 + y^2 = 2$

- 5 Find the equation of the tangent and the normal to the curves with these parametric equations at the point where t has the given value.

a $x = 2t - 1, \quad y = t^3 + 1$ when $t = 1$ **b** $x = t^3 - 1, \quad y = t^2 + t + 1$ when $t = 2$

c $x = 1 + t, \quad y = \frac{1}{1-t}$ when $t = 2$ **d** $x = 2\sin t, \quad y = \sin 2t$ when $t = \frac{\pi}{6}$

- 6 Find, in each case, the area between the x -axis and the curve between the two points defined by the given values of t or x .

a $x = t + 2$ from $t = 1$ to $t = 4$ **b** $x = 2t + 1$ from $x = 3$ to $x = 9$
 $y = 3t - 1$ $y = \sqrt{t}$

c $x = \ln t$ from $t = 2$ to $t = 3$ **d** $x = e^{-t}$ from $x = 1$ to $x = 2$
 $y = t \sin t$ $y = e^{2t} + 1$

- 7 The curve C is defined parametrically, for $0 \leq \theta \leq \pi$, by the equations

$$x = 3\cos \theta, \quad y = 3\cos 2\theta + 6$$

- a** Find $\frac{dy}{dx}$ in terms of θ .

Explain why the gradient at any point on the curve C is never greater than 4.

- b** Find the Cartesian equation of C and sketch the graph of C .

- 8** A curve has parametric equations $x = t^2$, $y = 2t$
 R is the point on the curve where $t = r$
- Show that the normal to the curve at point R has a gradient of $-r$.
 - If S is the point $(s^2, 2s)$, find the gradient of the chord RS in terms of r and s .
 - If the chord RS is normal to the curve at R , show that $r^2 + rs + 2 = 0$
 - At what point does the normal at the point $(9, 6)$ meet the curve again?

- 9** The trajectory of a cricket ball is given parametrically by the equations

$$x = 10t, \quad y = 2 + 10t - 5t^2$$

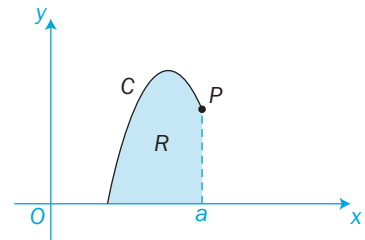
where x and y are the horizontal and vertical distances travelled (in metres) after a time of t seconds from the ball being struck.

- Find the Cartesian equation of the trajectory.
- Find the time taken before the ball hits the ground.
- What is the horizontal distance travelled by the ball before it hits the ground?

- 10** This diagram shows a sketch of part of the curve C with parametric equations

$$x = 1 + \frac{3}{t}, \quad y = t^2 \sin t, \quad \frac{\pi}{2} \leq t \leq \pi$$

- The point $P\left(a, \frac{\pi^2}{4}\right)$ lies on C . Find the value of a .
- Region R is enclosed by C , the x -axis and the line $x = a$ as shown in the diagram.



Show that the area of region R is given by $3 \int_{\frac{\pi}{2}}^{\pi} \sin t \, dt$

- Find the exact value of the area of R .

- 11** A curve has parametric equations

$$x = 2\cot \theta, \quad y = 2\sin^2 \theta, \quad 0 < \theta \leq \frac{\pi}{2}$$

- Find an expression for $\frac{dy}{dx}$ in terms of the parameter θ .
- Find an equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$
- Find a Cartesian equation of the curve in the form $y = f(x)$
 State the domain on which the curve is defined.

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Summary

Refer to

- You can sketch the graph of a curve given by the parametric equations $x = f(t)$, $y = g(t)$ by using a table of values showing the values of x and y as t varies. 7.1
- To convert the parametric equations $x = f(t)$, $y = g(t)$ of a curve into a Cartesian equation, eliminate the parameter t from the two equations. 7.1
- You can find the points of intersection of two curves by solving the equations of the curves simultaneously.

For example, you can substitute the parametric equations of one curve into the Cartesian equation of the other curve. 7.2

- To differentiate $x = f(t)$, $y = g(t)$ to find $\frac{dy}{dx}$,

the chain rule gives: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$ 7.3

- The area under a curve is given by $\int_{x=a}^{x=b} y dx = \int_{t=t_1}^{t=t_2} y \frac{dx}{dt} dt$

where the parameter t has the value t_1 at the point where $x = a$ and the value t_2 at the point where $x = b$ 7.4

Links

The path of any projectile is the result of two independent motions, horizontal and vertical, which can be expressed in terms of a parameter time t .

You can model the path of a projectile at any time t by the parametric equations

$$x(t) = (v_0 \cos \theta)t, \quad y(t) = (v_0 \sin \theta)t - \frac{1}{2}gt^2$$

where θ is the angle at which the projectile is launched at time $t = 0$, v_0 is the initial velocity of the projectile, and g is the acceleration due to gravity.

This model can be used to analyse the motion of a specific projectile, which is useful in areas such as sports science to study, for example, the flight of a golf ball.

