1 Hyperbolic functions

This chapter will show you how to

- use e^x to define hyperbolic functions
- solve hyperbolic equations
- prove identities involving hyperbolic expressions
- define inverse hyperbolic functions in terms of natural logarithms
- sketch the graphs of hyperbolic and inverse hyperbolic functions.

See Section 0.1.

See C3 for revision.

Before you start

You should know how to:

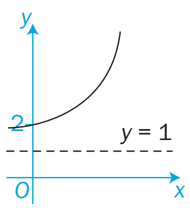
1 Solve an exponential equation.
e.g. Solve $4e^{2x} = 36$

Make the exponential term the subject:
$4e^{2x} = 36$ so $e^{2x} = 9$

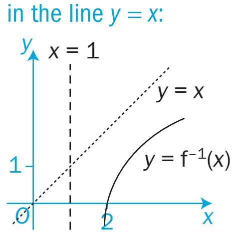
Take natural logs of both sides:
$2x = \ln 9$

$x = \frac{1}{2}\ln 9$, or $x = \ln 3$

2 Sketch the graph of $y = f^{-1}(x)$ for a given function f(x).
e.g. The graph of $y = f(x)$ is shown.



To sketch the graph of $y = f^{-1}(x)$, reflect the given graph and asymptote in the line $y = x$:



3 Prove a trigonometric identity.
e.g. Prove that $\sin 2\theta \sec\theta \equiv 2\sin\theta$

Use the identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$:

LHS $= \sin 2\theta \sec\theta$

$= (2\sin\theta\cos\theta) \times \frac{1}{\cos\theta}$

$= 2\sin\theta$

= RHS, as required

Check in:

1 Solve these equations. Give answers as natural logarithms.

a $e^{3x} - 1 = 7$

b $2e^{-x} = 8$

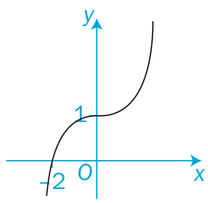
c $e^{2x} - 4e^x = 0$

d $e^{2x} - 8e^x + 15 = 0$

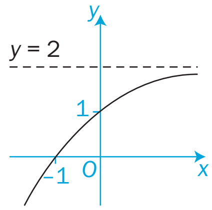
e $e^x - 9e^{-x} = 8$

2 For each graph $y = f(x)$ sketch, on a separate diagram, the graph of $y = f^{-1}(x)$

a



b



3 Prove these identities.

a $\frac{\cos 2\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$, for $\cos\theta \neq 0$

b $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$

c $\frac{1}{\sec\theta - 1} + \frac{1}{\sec\theta + 1} \equiv 2\cot\theta\operatorname{cosec}\theta$

1.1 Basic hyperbolic functions and their graphs

You can define the hyperbolic functions $\cosh x$ and $\sinh x$ in terms of the exponential function e^x.

For all real values of x, $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x$ is the hyperbolic cosine of x.
$\sinh x$ is pronounced 'shine x'.

EXAMPLE 1

Evaluate

a cosh 2 to 1 decimal place

b sinh (ln 2)

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a Use the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$:

$$\cosh 2 = \frac{1}{2}(e^2 + e^{-2})$$
$$= 3.762\ldots$$
$$= 3.8 \text{ to 1 d.p.}$$

Most scientific calculators have inbuilt hyperbolic functions – look for the HYP key.

b Use the definition $\sinh x = \frac{1}{2}(e^x - e^{-x})$:

$$\sinh(\ln 2) = \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right)$$
$$= \frac{1}{2}\left(2 - \frac{1}{2}\right)$$
$$= \frac{3}{4}$$

$$e^{-\ln 2} = \frac{1}{e^{\ln 2}} = \frac{1}{2}$$

You can define the hyperbolic tangent function, $\tanh x$ in terms of $\sinh x$ and $\cosh x$.

$\tanh x$ is pronounced 'than x'

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left(\frac{e^{2x} - 1}{e^x}\right)$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left(\frac{e^{2x} + 1}{e^x}\right)$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}$$

This definition resembles the result $\tan x \equiv \frac{\sin x}{\cos x}$, $\cos x \neq 0$ in trigonometry

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EXAMPLE 2

Evaluate $\tanh(\ln\sqrt{3})$

Use the exponential defintion of tanh *x*:

$$\tanh\left(\ln\sqrt{3}\right) = \frac{e^{2\ln\sqrt{3}} - 1}{e^{2\ln\sqrt{3}} + 1}$$

$$= \frac{3-1}{3+1}$$

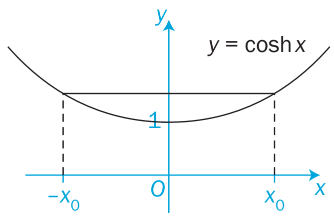
$$= \frac{1}{2}$$

$$e^{2\ln\sqrt{3}} = \left(e^{\ln\sqrt{3}}\right)^2 = \left(\sqrt{3}\right)^2 = 3$$

You can sketch the graph of hyperbolic functions by using properties of the exponential function e^x.

$$\cosh(-x_0) = \frac{1}{2}(e^{-x_0} + e^{-(-x_0)})$$

$$= \frac{1}{2}(e^{-x_0} + e^{x_0}) = \cosh(x_0)$$



so the graph of $y = \cosh x$ is symmetrical about the y-axis.

$$\cosh(0) = \frac{1}{2}(e^0 + e^{-0}) = \frac{1}{2}(1+1)$$

$$= 1$$

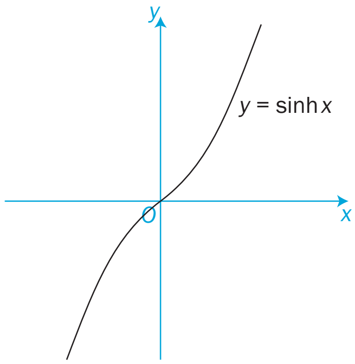
so the y-intercept of the graph is 1.

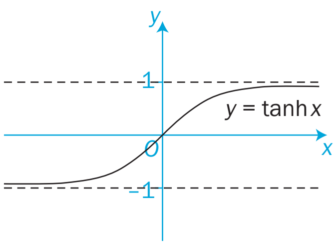
As $x \to \infty$, $e^x \to \infty$ and $e^{-x} \to 0$

that is, $\cosh x \to \infty$ as $x \to \infty$

$x \to \infty$ means x increases without limit

These diagrams show the graphs of $y = \sinh x$ and $y = \tanh x$:





The lines $y = 1$ and $y = -1$ are asymptotes to the graph $y = \tanh x$

$\cosh(-x) \equiv \cosh x$

The graph $y = \cosh x$ passes through $(0, 1)$ and is symmetrical about the y-axis.

$\cosh x \geqslant 1$ for all $x \in \mathbb{R}$

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$\sinh(-x) \equiv \sinh x$

The graph $y = \sinh x$ passes through $(0, 0)$ and has rotational symmetry, order 2, about the origin.

$\tanh(-x) \equiv -\tanh x$

The graph $y = \tanh x$ passes through $(0, 0)$ and has rotational symmetry, order 2, about the origin.
$|\tanh x| < 1$ for all $x \in \mathbb{R}$

$|\tanh x| < 1$ means $-1 < \tanh x < 1$

Exercise 1.1

1 Evaluate these expressions. Give each answer to 2 decimal places.

a $\cosh 3$ **b** $\sinh(-1)$

c $\tanh\left(\frac{1}{3}\right)$ **d** $\cosh(2)\sinh(3)$

2 Find the exact value of these expressions. Where appropriate, give answers in simplified surd form.

a $\sinh(\ln 3)$ **b** $\tanh(\ln 2)$

c $2\cosh(\ln\sqrt{2})$ **d** $\sinh\left(\ln\left(\sqrt{2}+1\right)\right)$

e $4\cosh\left(\ln\left(\sqrt{3}+1\right)\right)$ **f** $\tanh\left(\frac{1}{2}\ln\left(\sqrt{3}-1\right)\right)$

3 Evaluate these expressions. Leave simplified surds in your answers where appropriate.

a $\cosh(2x)$ when $x = \ln\left(1+\sqrt{2}\right)$

b $\sinh(y - \ln 2)$ when $y = \ln\left(4 - 2\sqrt{3}\right)$

c $\tanh\left(\frac{1}{2}z\right)$ when $z = \ln\left(1+\sqrt{3}\right)$

4 Use the exponential definitions of hyperbolic functions to prove these identities.

a $\cosh x + \sinh x \equiv e^x$

b $\dfrac{1+\tanh x}{1-\tanh x} \equiv e^{2x}$

c $\sinh(\ln x) \equiv \dfrac{(x+1)(x-1)}{2x}$, for $x > 0$

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5 On separate diagrams sketch the graphs of these equations. Mark any axis-crossing points with their exact coordinates and label any asymptotes with their equations.

a $y = 1 - \cosh x$

b $y = \sinh(x - \ln 2)$

c $y = \frac{1}{2}\tanh x + 1$

6 a Sketch, on the same diagram, the graph with equation $y = \sinh x$ and the graph with equation $y = 1 + \tanh x$

b Hence state the number of real roots of the equation

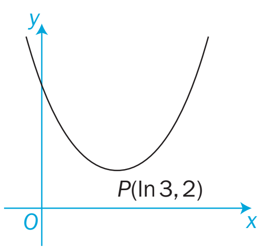
$$\sinh x - \tanh x - 1 = 0$$

c Use a similar technique to determine the number of real roots of these equations.

i $\cosh x - \tanh x = 0$

ii $\cosh x + \tanh x - 2 = 0$

7 The diagram shows the curve with equation $y = b\cosh(x + a)$ where a and b are constants. Point $P(\ln 3, 2)$ on this curve is a minimum point.



a State the exact value of a and the value of b.

b Find the y-coordinate of the point on this curve with x-coordinate $\ln 4$.

1.2 Reciprocal hyperbolic functions and their graphs

You can define the reciprocal hyperbolic functions $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\coth x$ in terms of the basic hyperbolic functions and also in terms of e^x.

For all real values of x, $\operatorname{sech} x \equiv \dfrac{1}{\cosh x} \equiv \dfrac{2}{e^x + e^{-x}}$

For all $x \in \mathbb{R}$, $x \neq 0$, $\operatorname{cosech} x \equiv \dfrac{1}{\sinh x} \equiv \dfrac{2}{e^x - e^{-x}}$

$$\coth x \equiv \frac{1}{\tanh x} \equiv \frac{e^{2x} + 1}{e^{2x} - 1}$$
$$\equiv \frac{\cosh x}{\sinh x}$$

EXAMPLE 1

a Given that $e^a = 4$ find the value of $\operatorname{sech} a$.

b Prove that $\coth x \operatorname{sech} x = \operatorname{cosech} x$ for all $x \neq 0$.

a Use the definition $\operatorname{sech} x \equiv \dfrac{2}{e^x + e^{-x}}$:

$$\operatorname{sech} a = \frac{2}{4 + \frac{1}{4}}$$
$$= \frac{8}{16 + 1} = \frac{8}{17}$$

Multiply the numerator and denominator by 4 to clear the fraction.

In this question it is not necessary to find a in order to evaluate $\operatorname{sech} a$.

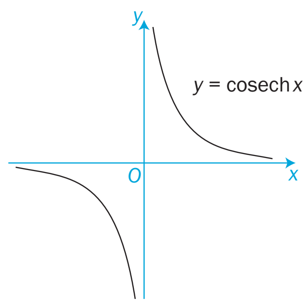
b Use the definitions $\coth x \equiv \dfrac{\cosh x}{\sinh x}$, $\operatorname{sech} x \equiv \dfrac{1}{\cosh x}$ and $\operatorname{cosech} x \equiv \dfrac{1}{\sinh x}$:

If $x \neq 0$ then $\coth x \operatorname{sech} x \equiv \dfrac{\cosh x}{\sinh x} \times \dfrac{1}{\cosh x}$

$$\equiv \frac{1}{\sinh x}$$
$$\equiv \operatorname{cosech} x$$

Cancel the $\cosh x$ terms.

You can use the properties of a basic hyperbolic function to sketch the corresponding reciprocal graph.



The diagram shows the graph with equation $y = \operatorname{cosech} x$.
$\sinh 0 = 0$, so $\operatorname{cosech} 0$ is undefined, that is the line $x = 0$ is an asymptote to the curve.

As $x \to \infty$, $\sinh x \to \infty$ and so $\dfrac{1}{\sinh x} \to 0$

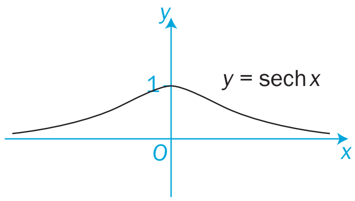
Refer to Section 1.1.

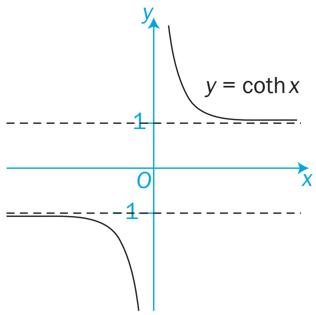
that is $\operatorname{cosech} x \to 0$ as $x \to \infty$

$$\operatorname{cosech}(-x) \equiv \frac{1}{\sinh(-x)} \equiv -\frac{1}{\sinh x} \equiv -\operatorname{cosech} x$$

so the graph $y = \operatorname{cosech} x$ has rotational symmetry, order 2, about the origin.
The line $y = 0$ is also an asymptote to the curve.

The graphs of the equations $y = \operatorname{sech} x$ and $y = \coth x$ are:





Exercise 1.2

1 Evaluate these expressions. Give each answer to 2 decimal places.

a $\operatorname{cosech} 2.5$ **b** $\operatorname{sech}\left(-\frac{1}{2}\right)$ **c** $\coth(\pi)$

2 Find the values of these expressions. Where appropriate, answers should be in simplified surd form.

a $\coth(\ln 5)$ **b** $\operatorname{sech}\left(\ln\sqrt{5}\right)$ **c** $2\operatorname{cosech}\left(\frac{1}{2}\ln 3\right)$

3 Use appropriate definitions to prove these identities.

Assume $x \neq 0$ where appropriate.

a $\operatorname{sech} x \equiv \dfrac{2e^x}{e^{2x}+1}$ **b** $\operatorname{sech} x \sinh x \coth x \equiv 1$

c $\operatorname{cosech} x + \operatorname{sech} x \equiv e^x \operatorname{cosech} x \operatorname{sech} x$ **d** $\dfrac{\operatorname{cosech} x}{\coth x - 1} \equiv e^x$

4 Given that $e^a = \frac{5}{2}$ find the value of

See Example 1.

a $\operatorname{sech} a$ **b** $\operatorname{cosech} a$ **c** $\coth a$

Give each answer as a fraction in its lowest terms.

5 Given that $e^{2b} = 9$ find the value of

a $\coth b$ **b** $\operatorname{sech}(2b)$ **c** $\operatorname{cosech} b$

Give each answer as a fraction in its lowest terms.

6 **a** Given that $5\operatorname{sech} a - 3\operatorname{cosech} a = 0$, for a a constant, find the value of $\tanh a$.

b Show that $e^{2a} = 4$ and hence find the exact value of a.

7 Sketch, on separate diagrams, the graphs with these equations. Label any asymptotes with their equations and any axis-crossing points with their exact coordinates.

a $y = 2\operatorname{sech} x + 1$ **b** $y = 3\operatorname{cosech}(x + \ln 2)$ **c** $y = -\coth(x - 1)$

FP3

1.3 Solving equations involving hyperbolic functions

You can solve an equation involving hyperbolic functions by using their exponential definitions.

EXAMPLE 1

Find the values of x for which

$5\cosh x + \sinh x = 7$

giving each answer as a natural logarithm.

Replace the hyperbolic functions in the equation with their exponential definitions:

$5\cosh x + \sinh x = 7$

so $\frac{5}{2}(e^{x} + e^{-x}) + \frac{1}{2}(e^{x} - e^{-x}) = 7$

$3e^{x} + 2e^{-x} = 7$

$\cosh x \equiv \frac{1}{2}(e^{x} + e^{-x})$, $\sinh x \equiv \frac{1}{2}(e^{x} - e^{-x})$

Collect like terms: $\frac{5}{2}e^{x} + \frac{1}{2}e^{x} = 3e^{x}$, $\frac{5}{2}e^{-x} - \frac{1}{2}e^{-x} = 2e^{-x}$

Multiply all terms in this equation by e^{x} to produce a quadratic equation:

$3(e^{x})^{2} + 2 = 7e^{x}$

$2e^{-x} \times e^{x} = 2e^{0} = 2$

$3(e^{x})^{2} - 7e^{x} + 2 = 0$

so $(3e^{x} - 1)(e^{x} - 2) = 0$

The quadratic equation is in the variable e^{x}

$e^{x} = \frac{1}{3}$ or $e^{x} = 2$

Hence the values of x are $x = \ln\left(\frac{1}{3}\right)$ or $x = \ln 2$

The answer $x = \ln\left(\frac{1}{3}\right)$ could also be written as $x = -\ln 3$

To solve an equation of the form $a\cosh x + b\sinh x = c$ where $a, b, c \in \mathbb{R}$, replace each hyperbolic function with its exponential form and solve, if possible, a quadratic in e^{x}.

Exercise 1.3

1 Solve these equations, giving each answer as a natural logarithm.

a $5\cosh x - \sinh x = 7$ **b** $4\cosh x + \sinh x = 8$ **c** $3\cosh x - 7\sinh x = 9$

2 Solve these equations, giving each answer correct to 2 decimal places.

a $4\cosh x + 3\sinh x = 3$ **b** $6\sinh x = 7\cosh x - 4$

FP3

3 Find the value of x for which

$$5\cosh x + 4\sinh x = 3$$

Give your answer as a natural logarithm.

4 Find the exact values of x which satisfy the equation

$$\frac{1}{2}\cosh x - \frac{1}{3}\sinh x = \frac{1}{2}$$

5 Find the non-zero value of x which satisfies the equation

$$\sqrt{3}\cosh x + \sqrt{2}\sinh x = \sqrt{3}$$

Give your answer in the form $\ln\left(a + b\sqrt{6}\right)$ for integers a and b.

6 It is given that the equation

$$7\cosh x + k\sinh x = 5$$

where k is a constant, is satisfied by the value $x = \ln 2$.

a Find the value of k.

b Hence find the other value of x which satisfies this equation. Give your answer as a natural logarithm.

7 **a** Find the exact value of x for which

$$5\cosh x + 9\sinh x = 13$$

$\cosh(-a) = \cosh a$
$\sinh(-a) = -\sinh a$

b Hence, or otherwise, find the solution of the equation

$$5\cosh\left(\frac{1}{2}x\right) - 9\sinh\left(\frac{1}{2}x\right) = 13$$

giving your answer in the form $\ln q$ for q a rational number in its lowest terms.

8 Solve these equations, giving answers as natural logarithms

a $7 + 4\tanh x = 17\operatorname{sech} x$ **b** $4\coth x - 7 = 16\operatorname{cosech} x$

9 Consider the equation

$$a\cosh x + b\sinh x = a$$

for non-zero constants a and b such that $a + b \neq 0$.

Prove that this equation has a non-zero real root exactly when

$$b^2 < a^2$$

In this case, write down this non-zero root in terms of a and b.

FP3

1.4 Further identities involving hyperbolic functions

You can prove standard identities involving hyperbolic functions by using their exponential definitions.

EXAMPLE 1

Prove that $2\sinh x \cosh x \equiv \sinh(2x)$

Replace $\sinh x$ and $\cosh x$ with their exponential definitions:

$$\text{LHS} \equiv 2\sinh x \cosh x$$
$$\equiv 2\left(\frac{1}{2}\left(e^{x} - e^{-x}\right)\right)\left(\frac{1}{2}\left(e^{x} + e^{-x}\right)\right)$$
$$\equiv \frac{1}{2}\left(\left(e^{x}\right)^2 - \left(e^{-x}\right)^2\right)$$
$$\equiv \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$
$$\equiv \sinh(2x) \equiv \text{RHS}$$

Hence $2\sinh x \cosh x \equiv \sinh(2x)$, as required.

By definition, $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$ and so $\sinh(2x) = \frac{1}{2}(e^{2x} - e^{-2x})$

This result is usually written as $\sinh 2x \equiv 2\sinh x \cosh x$

Some standard identities involving hyperbolic functions are:

$$\cosh^2 x - \sinh^2 x \equiv 1 \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$
$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x \qquad \cosh 2x \equiv 2\cosh^2 x - 1$$
$$\coth^2 x - 1 \equiv \operatorname{cosech}^2 x,\ (x \neq 0) \qquad \cosh 2x \equiv 1 + 2\sinh^2 x$$
$$\sinh 2x \equiv 2\sinh x \cosh x \qquad \tanh 2x \equiv \frac{2\tanh x}{1 + \tanh^2 x}$$

Learn these identities – only some of them appear in the formula book.

You can evaluate hyperbolic expressions by using an identity.

EXAMPLE 2

Given that $\sinh x = \sqrt{3}$ find the value of

a $\cosh x$ **b** $\tanh x$

a Replace $\sinh x$ with $\sqrt{3}$ in the identity $\cosh^2 x - \sinh^2 x \equiv 1$:

$$\cosh^2 x - \left(\sqrt{3}\right)^2 = 1$$
$$\cosh^2 x = 1 + 3$$
$$\cosh x = 2$$

$\cosh^2 x = 4$ so $\cosh x = \pm 2$, but $\cosh x > 0$ for all x, so $\cosh x = 2$

Refer to Section 1.1.

b Use the definition $\tanh x \equiv \frac{\sinh x}{\cosh x}$:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\sqrt{3}}{2}$$

Osborn's rule provides a link between the identities involving hyperbolic functions and the trigonometric functions.

See C3 for revision of trigonometric functions.

Osborn's rule:
- Start with a standard trigonometric identity.
- Replace each trigonometric function in the given identity with the corresponding hyperbolic function (where $\sin x$ corresponds to $\sinh x$ etc.).
- Change the sign whenever a replaced function involves a product of two sine functions.

For example, to apply Osborn's rule to the trigonometric identity

$$\cos 2x \equiv 1 - 2\sin^2 x$$

replace $\cos 2x$ with $\cosh 2x$ and

and $\sin^2 x$ with $-\sinh^2 x$

Osborn's rule then gives

$$\cosh 2x \equiv 1 + 2\sinh^2 x$$

which is one of the standard hyperbolic identities.

$\sin^2 x$ is the product of $\sin x$ with itself, so change the sign.

Exercise 1.4

1 Use definitions in terms of e^x to prove these identities.

a $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$

b $\sinh 3x \equiv \sinh x\left(3 + 4\sinh^2 x\right)$

c $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$

Take care when applying Osborn's rule – see question 12 in this exercise 1.4.

$\operatorname{sech}^2 x$ means $(\operatorname{sech} x)^2$

2 **a** Use definitions in terms of e^x to prove

$$\sinh(x + y) \equiv \sinh x \cosh y + \cosh x \sinh y$$

b Deduce that

$$\sinh 2A \equiv 2\sinh A \cosh A \quad \text{for all } A \in \mathbb{R}$$

3 Given that $\sinh x = 2$ use an appropriate identity to find the exact value of

a $\cosh x$ **b** $\sinh 2x$ **c** $\cosh 2x$

FP3

4 Given that $\cosh x = \frac{3}{2}$ use an appropriate identity to find the possible values of

a $\sinh x$ **b** $\tanh x$ **c** $\tanh 2x$

Give each answer in the form $a\sqrt{5}$ where a is a rational number in its lowest terms.

5 **a** Write down the result of applying Osborn's rule to these trigonometric identities.

i $\cos(x + y) \equiv \cos x \cos y - \sin x \sin y$

ii $\cos 3x \equiv 4\cos^3 x - 3\cos x$

iii $\tan 2x \equiv \dfrac{2\tan x}{1 - \tan^2 x}$

iv $\cot^2 x + 1 \equiv \operatorname{cosec}^2 x$

b Prove each identity found in part **a** by using definitions in terms of e^x.

6 It is given that $\tanh a = \frac{1}{2}\sqrt{3}$

a Using a suitable identity, show that $\cosh a = 2$

b Find the value of $\operatorname{cosech} 2a$. Give your answer in simplified surd form.

FP3

7 **a** Using only the identities given in the FP3 section of the formula booklet, prove that

$$\cosh 2x \equiv 1 + 2\sinh^2 x$$

b Hence show that

$$\frac{2}{(\cosh y) - 1} \equiv \operatorname{cosech}^2 \frac{1}{2}y \quad \text{for } y \neq 0$$

8 **a** By replacing $\cosh x$ and $\sinh x$ with their definitions in terms of e^x, prove that

$$(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv 1$$

b Find the exact value of x which satisfies the simultaneous equations.

$$\cosh x - \sinh^2 x = \frac{11}{16}$$

$$\cosh^2 x + \sinh x = \frac{37}{16}$$

Add the equations together.

9 If $\sinh x = \tan\theta$, where θ is an acute angle,

a show that $\cosh x = \sec\theta$,

b express $\tanh 2x$ in terms of $\sin\theta$.

10 Use any standard results to prove these identities.

a $\dfrac{2}{1 + \cosh 2x} \equiv \operatorname{sech}^2 x$

b $\coth x + \tanh x \equiv 2\coth 2x$

c $\sinh 2x \equiv \dfrac{2\tanh x}{1 - \tanh^2 x}$

11 **a** Prove that

$$e^{nx}\operatorname{sech} nx \equiv 1 + \tanh nx, \text{ for any integer } n$$

b Hence sketch the graph with equation

$$y = \frac{e^{3x}}{\cosh 3x}$$

Mark any axis-crossing points with their coordinates and label the equations of any asymptotes to this curve.

12 **a** Prove that

$$(\cos x + \sin x)^2 \equiv 1 + \sin 2x$$

b Show that applying Osborn's rule to this identity does *not* produce a valid identity involving hyperbolic functions. Justify your answer.

Osborn's rule may not work when applied to trigonometric functions which are nested inside a bracket.

13 **a** Find the possible values of $a \in \mathbb{R}$ for which

$$\left(\sqrt{2} + a\right)^4 = 17 + 12a\sqrt{2}$$

b Prove that

$$4\cosh^4 x + 4\sinh^4 x \equiv 3 + \cosh 4x$$

Express each of $\cosh^2 x$ and $\sinh^2 x$ in terms of $\cosh 2x$.

c Hence solve the equation

$$\cosh^4 x + \sinh^4 x = 5$$

Give each answer as a natural logarithm.

FP3

1.5 Inverse hyperbolic functions and their graphs

> The inverse function of $\sinh x$ is $\operatorname{arsinh} x$.
> $\operatorname{arsinh} x$ is the value y such that $\sinh y = x$

arsinh x is pronounced 'are shine x'.

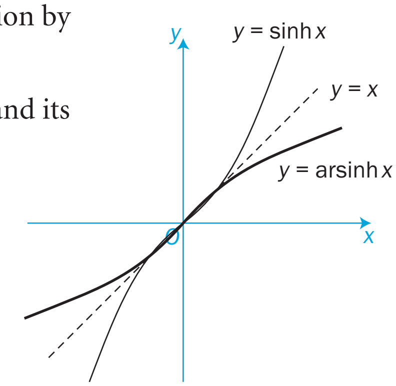
For example, $\sinh(\ln 2) = \frac{3}{4}$, so $\operatorname{arsinh}\left(\frac{3}{4}\right) = \ln 2$

$\sinh(\ln 2) = \frac{3}{4}$ Refer to Section 1.1, Example 1.

The inverse functions $\operatorname{arcosh} x$ and $\operatorname{artanh} x$ are similarly defined.

You can sketch the graph of an inverse hyperbolic function by using a reflection in the line $y = x$.

This diagram shows the graph with equation $y = \sinh x$ and its reflection (shown in bold) in the line $y = x$.
The graph in bold has equation $y = \operatorname{arsinh} x$



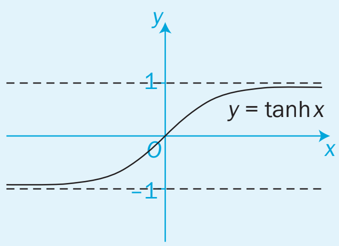
FP3

EXAMPLE 1

a Draw the graph of $y = \tanh x$
b Sketch the graph with equation $y = \operatorname{artanh} x$.
Clearly label the asymptotes to each graph.

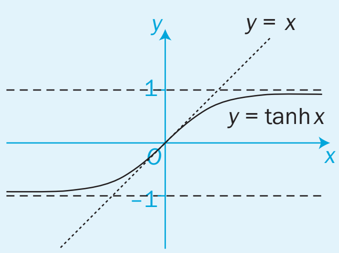
---

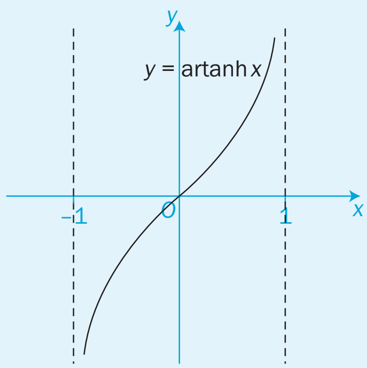
a



b Draw the line $y = x$ on a copy of the graph in part **a**.
Reflect this graph in the line.

Reflect the asymptotes in the line $y = x$



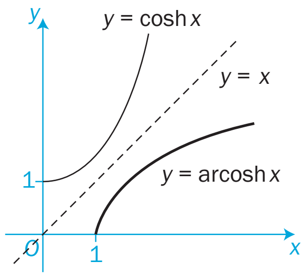


You can sketch the graph with equation $y = \text{arcosh}\, x$ where $x \in \mathbb{R}, x \geqslant 1$

Only one-to-one functions have inverses. Refer to C3.

In the diagram, the graph with equation $y = \cosh x$ for $x \in \mathbb{R}, x \geqslant 0$ displays a one-to-one function, with range $y \in \mathbb{R}, y \geqslant 1$.

Reflecting this graph in the line $y = x$ gives the graph with equation $y = \text{arcosh}\, x$ (shown in bold), which has domain $x \in \mathbb{R}, x \geqslant 1$ and range $y \in \mathbb{R}, y \geqslant 0$



It is conventional to restrict cosh x over $x \geqslant 0$ rather than $x \leqslant 0$ when defining its inverse.

arsinh x has domain and range $y \in \mathbb{R}$
arcosh x has domain $x \in \mathbb{R}, x \geqslant 1$ and range $y \in \mathbb{R}, y \geqslant 0$
artanh x has domain $x \in \mathbb{R}, -1 < x < 1$ and range $y \in \mathbb{R}$

The function arcosh x is only defined for $x \geqslant 1$

You can express the functions arsinh x, arcosh x and artanh x using natural logarithms.

EXAMPLE 2

Show that $\text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$

Define $y = \text{arsinh}\, x$ and use the definition of this inverse function:

$y = \text{arsinh}\, x$ so $\sinh y = x$

Use the result $\cosh^2 y - \sinh^2 y = 1$ to find $\cosh y$ in terms of x:

Refer to Section 1.4

$$\cosh^2 y = \sinh^2 y + 1$$
$$= x^2 + 1$$

So $\cosh y = \sqrt{x^2 + 1}$

$\cosh y > 0$ so ignore the negative square root $-\sqrt{x^2 + 1}$

Use the result $\sinh y + \cosh y = e^y$ to find y in terms of x:

$$\sinh y + \cosh y = \frac{1}{2}(e^y - e^{-y}) + \frac{1}{2}(e^y + e^{-y}) = e^y$$

So $x + \sqrt{x^2 + 1} = e^y$

Hence $y = \ln\left(x + \sqrt{x^2 + 1}\right)$

that is $\text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$ for $x \in \mathbb{R}$

$\text{arcosh}\, x = \ln\left(x + \sqrt{x^2 - 1}\right)$ for $x \in \mathbb{R}, x \geqslant 1$

$\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$ for $x \in \mathbb{R}, -1 < x < 1$

These results are in the FP3 section of the formula book. You may be asked to prove them in the exam.

You can use logarithmic forms to solve equations involving hyperbolic functions.

EXAMPLE 3

Solve the equation $\cosh x = 2$, giving answers as natural logarithms.

Use the result $\text{arcosh}\, x = \ln\left(x + \sqrt{x^2 - 1}\right)$:

$\cosh x = 2$ so one possible value of x is

$$\text{arcosh}\, 2 = \ln\left(2 + \sqrt{2^2 - 1}\right) = \ln\left(2 + \sqrt{3}\right)$$

Use the symmetry of the graph of $y = \cosh x$ in the y-axis to solve the equation:

The solution to the equation

$\cosh x = 2$ is $x = \pm\ln\left(2 + \sqrt{3}\right)$

The answers can also be written as $x = \ln\left(2 \pm \sqrt{3}\right)$

The equation $\cosh x = a$, where $a \geqslant 1$, has solution $x = \pm\ln\left(a + \sqrt{a^2 - 1}\right)$

This solution can also be written as $x = \ln\left(a \pm \sqrt{a^2 - 1}\right)$

FP3

Exercise 1.5

1 Use appropriate logarithmic forms to evaluate these expressions. Give answers as natural logarithms in as simple a form as possible.

a $\text{arsinh}\, 3$ **b** $\text{arcosh}\sqrt{2}$ **c** $2\,\text{artanh}\frac{1}{3}$

d $\text{arsinh}\left(\frac{4}{3}\right)$ **e** $\text{arcosh}\left(\frac{\sqrt{5}}{2}\right)$ **f** $\text{artanh}\frac{1}{\sqrt{2}}$

2 Sketch, on separate diagrams, the complete graphs with these equations. Use the largest possible domain for each graph. Mark any axis-crossing points with their exact values and any asymptotes with their equations.

a $y = 1 + \text{arcosh}(x - 2)$ **b** $y = -\text{artanh}\, 2x$ **c** $y = 2\,\text{arsinh}\left(x + \sqrt{3}\right)$

3 Solve these equations. Give answers in form $\ln\left(a + b\sqrt{c}\right)$, for $a, b \in \mathbb{R}$ and c a prime.

a $\sinh x = 4$ **b** $\cosh x = 3$ **c** $\sinh\frac{1}{2}x = 2$

d $\sqrt{3}\tanh\frac{1}{2}x = 1$ **e** $2\tanh x + \sqrt{3} = 0$ **f** $\cosh 2x = 3$

For part **f** express $3 + 2\sqrt{2}$ in the form $\left(a + \sqrt{b}\right)^2$ where $a, b \in \mathbb{R}$.

4 **a** Given that $y = \text{artanh}\, x$, where $x \in \mathbb{R}$, $-1 < x < 1$, show that $x = \dfrac{e^{2y} - 1}{e^{2y} + 1}$

b Deduce that $\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$

$f(x) = 2\,\text{artanh}\left(\frac{x - 1}{x}\right) x \in \mathbb{R}, x > \frac{1}{2}$

c Show that $f(x) = \ln(ax + b)$, stating the value of the constants a and b.

5 Given that $\cosh x = a$, where $a \in \mathbb{R}, a \geqslant 1$

a show that $e^{2x} - 2ae^{x} + 1 = 0$

b Deduce that the equation $\cosh x = a$ has solutions given by $x = \ln\left(a \pm \sqrt{a^2 - 1}\right)$

6 Solve these equations. Give answers in terms of natural logarithms.

a $\sinh^2 x = 2$

b $8\cosh^3 x = 27$

c $\sqrt{5\tanh x} = 2$

d $\operatorname{sech}^2 \frac{1}{2}x = \frac{1}{2}$

7 **a** Use the definition of $\sinh y$ in terms of exponentials to show that

if $x = \sinh y$ then $e^{2y} - 2xe^{y} - 1 = 0$

b Hence, or otherwise, show that $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

8 Use logarithmic forms to prove that

a $\operatorname{arsinh} x + \operatorname{arsinh}(-x) = 0, x \in \mathbb{R}$

b $\operatorname{artanh} x + \operatorname{artanh}(-x) = 0, -1 < x < 1.$

9 Given that $\tanh a = -\frac{1}{7}$

a find the exact value of a, giving your answer in the form $\ln k$.

b hence, or otherwise, show that $\sinh a = -\frac{1}{12}\sqrt{3}$

10 Given that $\sinh b = \sqrt{8}$

a find the exact value of b

b hence, or otherwise, find the value of $\tanh b$, giving your answer in the form $k\sqrt{2}$, where k is a rational number in its lowest terms.

11 **a** Simplify the expression $\operatorname{arsinh}\left(\frac{1}{x}\right) + \ln x$, where $x > 0$, giving your answer as the natural logarithm of a function.

b Hence solve, in terms of e, the equation $\operatorname{arsinh}\left(\frac{1}{x}\right) + \ln x = 1$

12 Show that

a $\operatorname{arsinh}\left(\sqrt{x^2 - 1}\right) \equiv \operatorname{arcosh} x$ for $x \geqslant 1$

b $\operatorname{artanh}(1 - 2x) \equiv \frac{1}{2}\ln\left(\frac{1 - x}{x}\right)$ for $0 < x < 1$

13 Use any appropriate identities to show that

$\operatorname{artanh}(\operatorname{sech} x) \equiv \ln(\coth x + \operatorname{cosech} x)$ for all $x > 0$

FP3

1.6 Further equation solving

You can use an identity to solve an equation involving hyperbolic expressions.

EXAMPLE 1

Solve the equation $\sinh^2 x - 3\cosh x + 3 = 0$
giving answers as natural logarithms where appropriate.

Use the identity $\cosh^2 x - \sinh^2 x \equiv 1$ to rewrite the given equation in terms of $\cosh x$:

$\sinh^2 x - 3\cosh x + 3 = 0$ so $(\cosh^2 x - 1) - 3\cosh x + 3 = 0$

so $\cosh^2 x - 3\cosh x + 2 = 0$

hence $(\cosh x - 1)(\cosh x - 2) = 0$

$\sinh^2 x = \cosh^2 x - 1$
Refer to Section 1.4

So either $\cosh x = 1$ or $\cosh x = 2$

If $\cosh x = 1$ then $x = 0$

You can see from the graph of $y = \cosh x$ that the equation $\cosh x = 1$ is satisfied only by $x = 0$

If $\cosh x = 2$ then $x = \ln\left(2 \pm \sqrt{2^2 - 1}\right)$

$= \ln\left(2 \pm \sqrt{3}\right)$

If $\cosh x = a$ then
$x = \ln\left(a \pm \sqrt{a^2 - 1}\right)$
Refer to Section 1.5.

Hence the solution to the equation $\sinh^2 x - 3\cosh x + 3 = 0$ is $x = 0$, $x = \ln\left(2 \pm \sqrt{3}\right)$

EXAMPLE 2

Find the values of x for which
$2\sinh^2 x = \sinh x \cosh x$
giving answers as natural logarithms where appropriate.

Gather terms to one side:

$2\sinh^2 x = \sinh x \cosh x$ so $2\sinh^2 x - \sinh x \cosh x = 0$

that is, $\sinh x\,(2\sinh x - \cosh x) = 0$

Avoid dividing through by $\sinh x$ which might be zero.

Hence, either $\sinh x = 0$ or $2\sinh x - \cosh x = 0$

If $\sinh x = 0$ then $x = 0$

$\sinh x = 0$ only when $x = 0$ Refer to the graph of $y = \sinh x$

If $2\sinh x - \cosh x = 0$

$$2\sinh x = \cosh x$$

$$\frac{2\sinh x}{\cosh x} = 1$$

$$2\tanh x = 1$$

$$\tanh x = \frac{1}{2}$$

You can divide by $\cosh x$ because $\cosh x$ has a minimum value of 1. see Section 1.1.

FP3

EXAMPLE 2 (CONT.)

Use the result $\text{artanh}\,x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$:

$\tanh x = \frac{1}{2}$ so $x = \text{artan}\frac{1}{2}$

$$= \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{2+1}{2-1}\right)$$

$$= \frac{1}{2}\ln 3$$

Multiply all terms in the bracket by 2 to clear the fractions.

Hence the equation $2\sinh^2 x = \sinh x\cosh x$ has solutions $x = 0$, $x = \frac{1}{2}\ln 3$

Exercise 1.6

Unless told otherwise, give each solution in terms of natural logarithms.

1 Solve these equations.

a $\cosh^2 x - \sinh x - 7 = 0$

b $2\cosh^2 x + 5\sinh x + 1 = 0$

c $2\sinh^2 x - 5\cosh x + 5 = 0$

d $3\sinh^2 x + 3 = 7\cosh x$

2 Solve these equations.

a $4\sinh^2 x = 3\sinh x\cosh x$

b $3\sinh x\cosh x + 2\sinh^2 x = -2$

c $2\sinh^2 x + \sinh x\cosh x = 2\sinh x + \cosh x$

3 Solve the equation $\cosh x - 3\sinh x = 0$

a by using the definition $\tanh x \equiv \frac{\sinh x}{\cosh x}$

b by using the exponential definitions of $\cosh x$ and $\sinh x$.

4 Use suitable identities to solve these equations

a $2\sinh x\cosh x = 3$

b $\cosh^2\frac{3}{2}x + \sinh^2\frac{3}{2}x = \sqrt{2}$

c $\sinh 2x\cosh 2x = \sqrt{2}$

5 **a** Show that the equation $3\sinh^2 x = \sinh 2x$ can be expressed as $\sinh x(3\sinh x - 2\cosh x) = 0$

b Hence find the values of x for which $3\sinh^2 x = \sinh 2x$

FP3

6 Use any appropriate identities to solve these equations.

a $\cosh 2x - 7\cosh x + 6 = 0$ **b** $1 + \cosh 2x = 5\sinh x$

c $6\sinh^2 x + 3 = 8\sinh x\cosh x$ **d** $\sinh 2x + \cosh x = (\sinh x + 1)^2$

7 Use the identity $1 - \tanh^2 x = \text{sech}^2 x$ to solve these equations.

a $2\text{sech}^2 x + \tanh x - 2 = 0$ **b** $2\tanh^2 x + \text{sech}\, x = 2$

c $4\tanh^2 x = 5(1 - \text{sech}\, x)$ **d** $2\tanh^4 x + 3\text{sech}^2 x - 2 = 0$

8 **a** Simplify $\dfrac{1}{1-\tanh x} + \dfrac{1}{1+\tanh x}$

b Hence, or otherwise, solve the equation $\dfrac{1}{1-\tanh x} + \dfrac{1}{1+\tanh x} = 4$

Give answers as natural logarithms.

9 **a** Use the result $\sinh(A + B) \equiv \sinh A\cosh B + \cosh A\sinh B$ to find an identity for $\sinh 3x$ in terms of $\sinh x$.

b Hence, or otherwise, find the value of x for which

$8\sinh^3 x + 6\sinh x + \cosh 3x = 0$

Give your answer in the form $\frac{1}{a}\ln b$, where a and b are integers to be stated.

10 **a** Show that the equation $2\tanh 2x = \coth x$, where $x > 0$, can be re-arranged into the equation $3\tanh^2 x = 1$

b Hence, or otherwise, find the solution to the equation $2\tanh 2x = \coth x$.

Give your answer in the form $\ln\left(\dfrac{\sqrt{2}+\sqrt{a}}{2}\right)$, for a an integer to be stated.

11 **a** Use a suitable identity to show that the equation

$3 + 3\tanh^2 x = 8\tanh x$

can be expressed in the form

$\tanh 2x = k$, stating the value of the constant k.

b Hence, or otherwise, find the solution to the equation

$3 + 3\tanh^2 x = 8\tanh x$

FP3

12 **a** On the same diagram, sketch the graph with equation $y = \cosh x$ and the graph with equation $y = 1 + \operatorname{sech} 2x$

b Hence state the number of solutions to the equation $1 + \operatorname{sech} 2x = \cosh x$

P and Q are the a points where the curves $y = \cosh x$ and $y = 1 + \operatorname{sech} 2x$ intersect.

c Prove that the line PQ is horizontal and find its length. Give your answer to 2 decimal places.

13 Solve the equation $\cosh 4x = 16 \sinh x + 1$. When not exact, give your answer as a natural logarithm.

14 **a** Express $\cosh^4 x - \sinh^4 x$ as a single hyperbolic function.

b Hence, or otherwise, solve the equation

$\cosh^4 x - \sinh^4 x = \tanh x + \coth x$, where $x \neq 0$

Give your answer in exact form.

15 Solve these equations. Give your answers as natural logarithms.

a $\sinh^3 x + 7 = 3 \cosh^2 x$

b $\sinh^2 x + 4 \tanh^2 x = 3$

c $2 - \operatorname{sech}^2 x = \coth 2x$, $x \neq 0$

FP3

Review 1

1 Solve the equation

$$7\cosh x - 3\sinh x = 11$$

Give your answer as a natural logarithm.

2 It is given that the equation $5\tanh x - a\operatorname{sech} x = a$, where a is a constant, is satisfied by the value $x = \ln\left(\frac{7}{3}\right)$

a Find the value of a.

b Show that no other real value of x satisfies this equation.

3 a Use the definitions of cosh and sinh in terms of exponentials to prove that

$$\tanh(x + y) \equiv \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

b Given that $\tanh x = \frac{a-1}{a+1}$ where $a > 0$ is a constant, show that

$$\tanh(x + \ln 2) = \frac{4a - 1}{4a + 1}$$

4 a Using the exponential definition of $\cosh x$ prove that

$$\cosh 2x \equiv 2\cosh^2 x - 1$$

b Deduce that

$$\cosh 4x \equiv 8\cosh^4 x - 8\cosh^2 x + 1$$

c Hence, or otherwise, solve the equation

$$\cosh^4 x - \cosh^2 x = \frac{9}{64}$$

Give each answer in the form $q\ln 2$ for q a rational number in its lowest terms.

5 Solve the equation

$$\sinh 2x + \cosh x = \coth x, \text{ where } x \neq 0.$$

Give answers as natural logarithms.

6 Given that $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

a show that, for $x > 0$ $\operatorname{arcosech} x = \ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$

The graph with equation $y = \operatorname{arcosech} x$ intersects the graph with equation $y = \ln x$ at point P.

arcosech x is the inverse function of cosech x

b Find the exact coordinates of P.

FP3

7 **a** On the same diagram, sketch the graphs with equations

i $y = 3e^{-x}$

ii $y = 2\tanh\frac{1}{2}x$

Label axis crossing points with their coordinates and the asymptotes with their equations.

b Find the exact coordinates of the point where these graphs intersect.

8 Solve the equation $6\tanh^2 x + 5\operatorname{sech} x - 7 = 0$

Give your answers as natural logarithms.

9 Prove these identities. You may use any standard results in your proofs.

a $\dfrac{1}{\coth x - 1} + \dfrac{1}{\coth x + 1} \equiv \sinh 2x$

b $\dfrac{1}{1 - \tanh x} - \dfrac{\tanh x}{1 + \tanh x} \equiv \cosh 2x$

10 Solve the equation

$$2\coth x - 3\operatorname{cosech} x = 2$$

Give your answer in the form $\ln q$ for q a rational number in its lowest terms.

11 Given that

$$\cosh 2x + \sinh ax - 7 = 0$$

where $a \in \mathbb{R}$, solve this equation when

a $a = 1$ **b** $a = 2$

Give answers as natural logarithms.

12 **a** Prove that

$$(\operatorname{sech} x - \cosh x)^2 \equiv \sinh^2 x - \tanh^2 x$$

b Hence, or otherwise, solve the equation

$$\sinh^2 x + 3\tanh^2 x = 4$$

Give your answers as natural logarithms.

FP3

1 Exit

Summary

Refer to

- You can use the exponential function e^x to define hyperbolic functions:
  - $\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1}$
  - $\cosh(-x) \equiv \cosh x$, $\sinh(-x) \equiv -\sinh x$
    $\tanh(-x) \equiv -\tanh x$, $|\tanh x| < 1$ for all $x \in \mathbb{R}$ 1.1
  - $\operatorname{sech} x \equiv \frac{1}{\cosh x} \equiv \frac{2}{e^x + e^{-x}}$
  - $\operatorname{cosech} x \equiv \frac{1}{\sinh x} \equiv \frac{2}{e^x - e^{-x}} \quad (x \neq 0)$
  - $\coth x \equiv \frac{1}{\tanh x} \equiv \frac{\cosh x}{\sinh x} \equiv \frac{e^{2x} + 1}{e^{2x} - 1} \quad (x \neq 0)$ 1.2
- You can use the exponential definitions of hyperbolic functions to establish these identities:
  - $\cosh^2 x - \sinh^2 x \equiv 1$
  - $\sinh 2x \equiv 2\sinh x \cosh x$
  - $\cosh 2x \equiv \cosh^2 x + \sinh^2 x \equiv 2\cosh^2 x - 1 \equiv 1 + 2\sinh^2 x$
  - $\tanh 2x \equiv \frac{2\tanh x}{1 + \tanh^2 x}$
  - $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$
  - $\coth^2 x - 1 \equiv \operatorname{cosech}^2 x, \ (x \neq 0)$ 1.4
- You can use identities to show inverse hyperbolic functions have logarithmic equivalents.
  - $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ has domain $x \in \mathbb{R}$ and range $y \in \mathbb{R}$
  - $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$ has domain $x \in \mathbb{R}, x \geqslant 1$ and range $y \in \mathbb{R}, y \geqslant 0$
  - $\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$ has domain $x \in \mathbb{R}, -1 < x < 1$ and range $y \in \mathbb{R}$ 1.5

FP3

Links