TOPIC 1

Sequences and series – arithmetic

What do you need to know?

An arithmetic sequence is a series of numbers that go up or down by the same amount each time. This same amount that you *add* or *subtract* is called d, the *common difference*.

How do you do it?

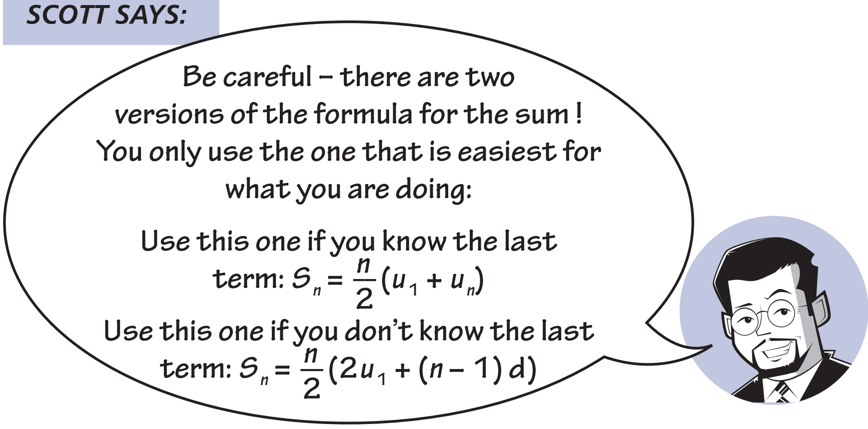
Step 1: Make sure that it's an arithmetic sequence

If it goes up or down by d each time, it's an arithmetic sequence: 7, 10, 13, 16,19, 22, …
If it doesn't do this, it's not an arithmetic sequence: 7, 9, 13, 20, 32, …

Step 2: Go to your formula booklet and get the formulae

2.5	The n^{th} term of an arithmetic sequence	$u_n = u_1 + (n - 1)\,d$
	The sum of n terms of an arithmetic sequence	$S_n = \frac{n}{2}(2u_1 + (n-1)\,d) = \frac{n}{2}(u_1 + u_n)$

SCOTT SAYS:



Step 3: Work out what you know and what is wanted

u_n = last term in sequence

u_1 = first term in sequence

n = number of terms in sequence

d = common difference

S_n = sum of the series

Step 4: Substitute and solve for what is wanted using algebra

Examples

- 3, 7, 11, 15, 19 ⇒ goes up by 4 each time – "common difference" is 4 ⇒ $d = 4$.
  To find the 32nd term, you would use the first formula:

  u_{32} = 3+(32-1).4=3+124=127.

  To find the sum of the first 32 terms of the series, you would do

  $S_{32} = \frac{32}{2}(3 + 127) = 2210.$
- 30, 22, 14, 6, – 2 ⇒ goes down by 8 each time – "common difference" is – 8 ⇒ $d = -8$.

To find the sum of the first 19 terms of the series without knowing the 19th term exactly, you would do.

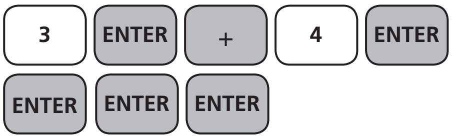
$S_{19} = \frac{19}{2}(2 \cdot 30 + (19 - 1) \cdot -8) = 9.5(60 - 144) = 9.5 \cdot -84 = -798.$

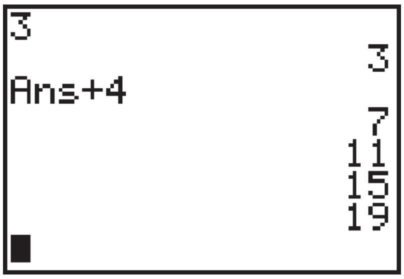
- If a sequence has a 5th term = 17 and a common difference of 6, you would use the first formula to find the 1st term: $17 = u_1 + (17 - 1).6 \Rightarrow 17u_1 + 96 \Rightarrow u_1 = -79$.

How can my calculator help me do this?

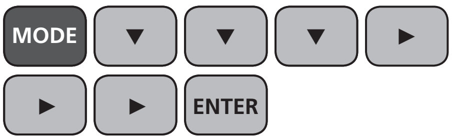
There are two things you can do to make your life easier.
Let's create one of the examples above using our GDC:

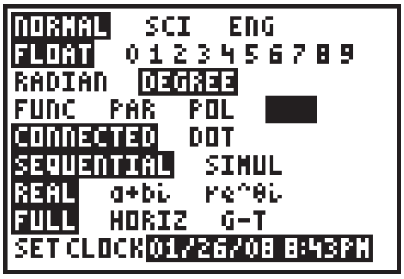
1st method

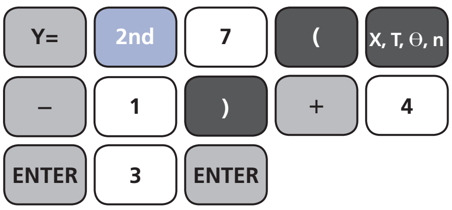


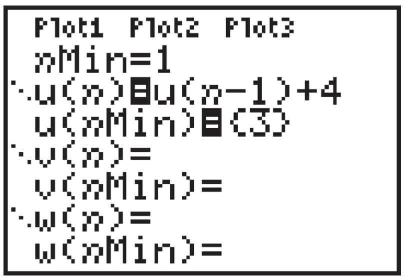


2nd method (trickier but great if you can master it!)

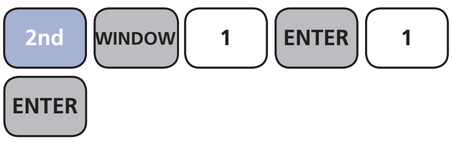


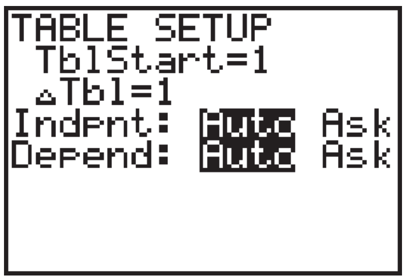




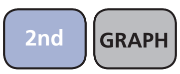


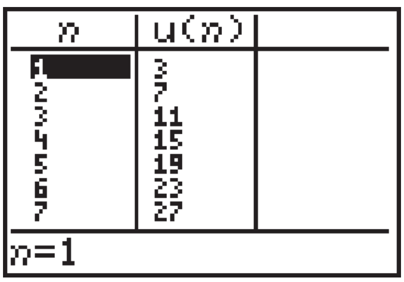
nMin means starting counting with the 1st term
u(n−1) means get the previous term
u(nMin) is the same as u_1 in our formula





TblStart is what term you want the calculator to start showing
ΔTbl is how much to go up by (usually 1 with these problems)





Examples from past IB papers

An easy one from May 1996 Paper 1

The cost of boring a well 300 metres deep is calculated from the following information:

The cost for the first metre is \$20.00, and then the cost per metre increases by \$2.00 for every subsequent metre.

Find

(a) the cost of boring the 300th metre;

(b) the total cost of boring the well.

Working:

(a) $u_1 = 20$

$u_2 = 22$

$u_{300} = 20 + (300 - 1)(2)$

$= 618$

(b) $S_{300} = \frac{300}{2}(20 + 618)$

$= 95\,700$

Answers:

(a) \$618

(b) \$95 700

A harder one from May 1998 Paper 1

A bank clerk is offered a yearly salary of CHF18 000 (Swiss Francs) for the first year of employment. She is also to receive CHF500 a year increase at the beginning of each subsequent year.

(a) What is her salary at the beginning of the fifth year?

After the first five years she is to receive a yearly increase of CHF750 until she reaches a maximum annual salary of CHF26 000.

(b) What is her salary at the beginning of the sixth year?

(c) At the beginning of which year of her employment does her salary reach the maximum?

Working:

(a) $18\,000 + 500(4) = 20\,000$

(b) $20\,000 + 750 = 20\,750$

(c) $\frac{(26\,000 - 20\,750)}{750} = 7$

$6 + 7 = 13$

Answers:

(a) CHF 20 000

(b) CHF 20 750

(c) 13th year

Now you practise it

An easy one from May 1999

A woman deposits $100 into her son's savings account on his first birthday. On his second birthday she deposits $125, $150 on his third birthday, and so on.

(a) How much money would she deposit into her son's account on his 17^{th} birthday?

(b) How much in total would she have deposited after her son's 17^{th} birthday?

Working:

Answers:

(a) ....

(b) ....

A harder one from November 2003

The fourth term of an arithmetic sequence is 12 and the tenth term is 42.

(a) Given that the first term is u_1 and the common difference is d, write down two equations in u_1 and d that satisfy this information.

(b) Solve the equations to find the values of u_1 and d.

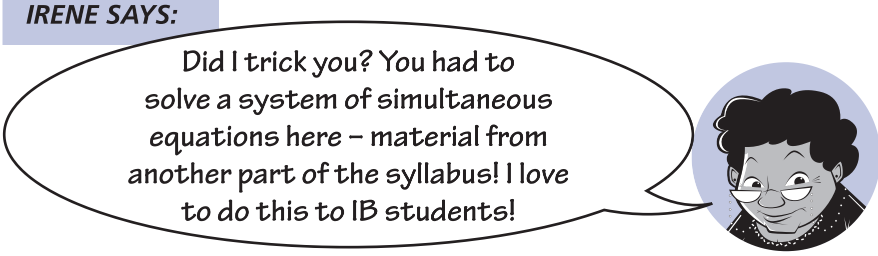
Working:

Answers:

(a) ....

(b) ....

IRENE SAYS:



Sequences and series – geometric

What do you need to know?

A geometric sequence is a series of numbers that go up or down by the same rate each time. This same amount that you *multiply* or *divide* by is called r, the *common ratio*.

How do you do it?

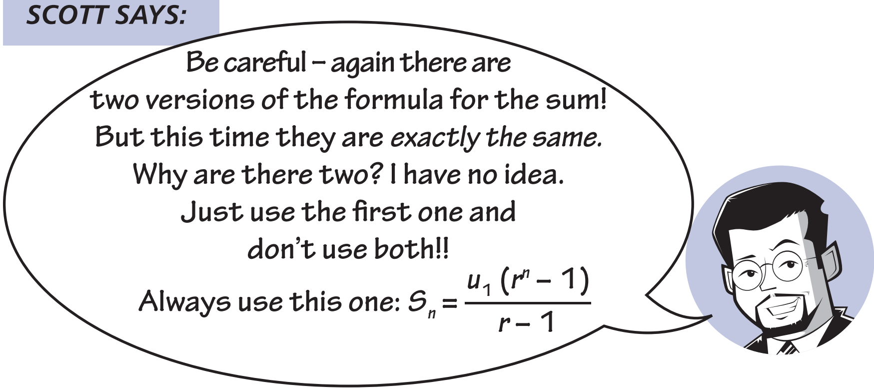
Step 1: Make sure that it's a geometric sequence

If it goes up or down by multiplying by r each time, it's a geometric sequence: 3, 12, 48, 192, 768, ... (multiply by 4)
If it doesn't do this, it's not a geometric sequence: 3, 12, 24, 96, 192, …

Step 2: Go to your formula booklet and get the formulae

2.6	The n^{th} term of a geometric sequence	$u_n = u_1 r^{n-1}$
	The sum of n terms of a geometric sequence	$S_n = \dfrac{u_1(r^n - 1)}{r - 1} = \dfrac{u_1(1 - r^n)}{1 - r}, r \neq 1$

SCOTT SAYS:



Step 3: Work out what you know and what is wanted

u_n = last term in sequence
u_1 = first term in sequence
n = number of terms in sequence
r = common ratio
S_n = sum of the series

Step 4: Substitute and solve for what is wanted using algebra.

Examples

- 3, 12, 48, 192, 768 $\Rightarrow$ multiply by 4 each time – "common ratio" is 4 $\Rightarrow r = 4$.
  To find the 12th term of the sequence, you would use the first formula:
  $u_{32} = 3 \cdot 4^{12-1} = 3 \cdot 4194304 = 12582912.$
  To find the sum of the first eight terms, you would use the second formula:
  $S_8 = \dfrac{3 \cdot (4^{8-1)}}{4 - 1} = \dfrac{3 \cdot 65535}{3} = 65535.$
- 40, 8, 1.6, 0.32, 0.064 $\Rightarrow$ multiply by 0.2 each time – "common ratio" is 0.2 $\Rightarrow r = 0.2$.
  To find the sum of the first eleven terms of the series, you would use the second formula again:
  $S_{11} = \dfrac{40 \cdot (0.2^{11-1)}}{0.2 - 1} = \dfrac{40 \cdot -0.99999997952}{-0.8} = 49.999998976 \approx 50.0 \text{ (3sf)}$

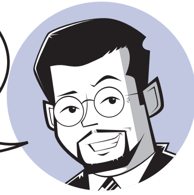
How can my calculator help me do this?

There are two things you can do to make your life easier.

Let's create one of the examples above using our GDC:

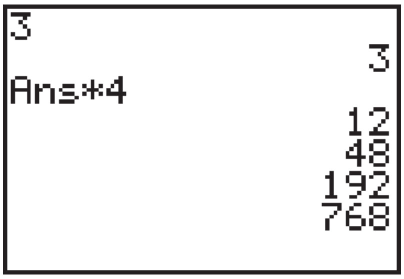
SCOTT SAYS:

Be careful – in that last example, you might want to divide by 5 each time instead of multiplying by 0.2. It's the same thing but with geometric sequences we like to multiply in order to find the "common ratio".



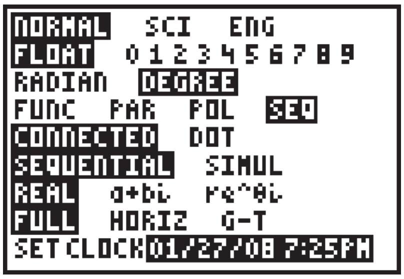
1st method

3 | ENTER | X | 4 | ENTER | ENTER | ENTER | ENTER

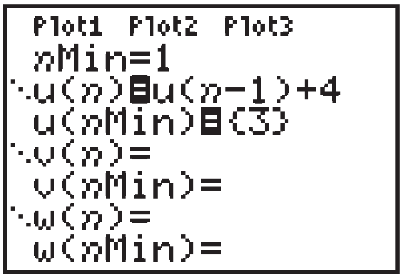


2nd method (trickier but great if you can master it!)

MODE | ▼ | ▼ | ▼ | ▶ | ▶ | ▶ | ▶ | ENTER



Y= | 2nd | 7 | (| X, T, Θ, n | – | 1 |) | X | 4 | ENTER | 3 | ENTER

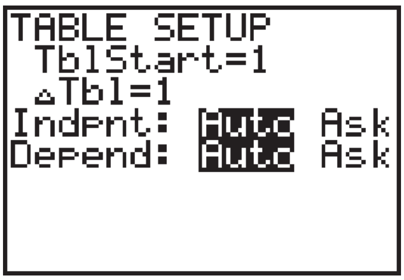


nMin means starting counting with the 1st term

u(n−1) means get the previous term

u(nMin) is the same as u_1 in our formula

2nd | WINDOW | 1 | ENTER | 1 | ENTER



TblStart is what term you want the calculator to start showing

ΔTbl is how much to go up by (usually 1 with these problems)

2nd | GRAPH

n	u(n)
1	3
2	12
3	48
4	192
5	768
6	3072
7	12288

n=1

Examples from past IB papers

An easy one from Specimen 2000 Paper 1

The tuition fees for the first three years of high school are given in the table below.

Year	Tuition fees (in dollars)
1	2 000
2	2 500
3	3 125

These tuition fees form a geometric sequence.

(a) Find the common ratio, r, for this sequence.

(b) If fees continue to rise at the same rate, calculate (to the nearest dollar) the total cost of tuition fees for the first six years of high school.

Working:

(a) $r = \frac{2500}{2000} = 1.25$

(b) $S_n = \frac{u_1(r^n - 1)}{r - 1}$

$S_6 = \frac{2000(1.25^6 - 1)}{1.25 - 1}$

$= 22\ 517.57813$

Answers:

(a) 1.25

(b) \$22 518

A harder one from May 2004 Paper 2

A basketball is dropped vertically. It reaches a height of 2 m on the first bounce. The height of each subsequent bounce is 90% of the previous bounce.

(a) What height does it reach on the 8th bounce?

(b) What is the total vertical distance travelled by the ball between the first and the sixth time the ball hits the ground?

Working:

(a) $u_n = 2(0.9)^7 = 0.9565938$

(b) $S_n = \frac{2((0.9)^5 - 1)}{(0.9 - 1)} = 8.1902$

Total distance travelled = 2×8.1902

$= 16.3804$ m

Answers:

(a) 0.957 m

(b) 16.4 m

Now you practise it

An easy one from Specimen 2005 Paper 1

A geometric sequence has all its terms positive. The first term is 7 and the third term is 28.

(a) Find the common ratio.

(b) Find the sum of the first 14 terms.

Working:

Answers:

(a) ....

(b) ....

A harder one from May 2000

The population of Bangor is growing each year. At the end of 1996, the population was 40 000. At the end of 1998, the population was 44 100. Assuming that these annual figures follow a geometric progression, calculate

(a) the population of Bangor at the end of 1997;

(b) the population of Bangor at the end of 1992.

Working:

Answers:

(a) ....

(b) ....

HANA AND CHRIS SAY:

A good trick here is to make u_1 represent the year 1992 and go up from there!